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Generalized dipole polarizabilities and the spatial structure of hadrons

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Abstract

We present a phenomenological discussion of spin-independent, generalized dipole polarizabilities of hadrons entering the virtual Compton scattering process $\gamma^* h \to \gamma h$. We introduce a new method of obtaining a tensor basis with appropriate Lorentz-invariant amplitudes which are free from kinematical singularities and constraints. The result is summarized in terms of a compact effective Lagrangian. We then motivate a gauge-invariant separation into a generalized Born term containing ground-state properties only, and a residual contribution describing the model-dependent internal structure. The generalized dipole polarizabilities are defined in terms of Lorentz-invariant residual amplitudes. Particular emphasis is laid on a physical interpretation of these quantities as characterizing the spatial distributions of the induced electric polarization and magnetization of hadrons. It is argued that three dipole polarizabilities, namely the longitudinal electric $\alpha_L(q^2)$, the transverse electric $\alpha_T(q^2)$, and the magnetic $\beta(q^2)$ ones are required in order to fully reconstruct local polarizations induced by soft external fields in a hadron. One of these polarizabilities, $\alpha_T(q^2)$, describes an effect of higher order in the soft final-photon momentum q'. We argue that the associated spatial distributions obtained via the Fourier transforms in the Breit frame are meaningful even for such a light particle as the pion. The spatial distributions are determined at large distances $r \sim 1/m_\pi$ for pions, kaons, and octet baryons by use of ChPT.

I. INTRODUCTION

Recently, Compton scattering with virtual photons (virtual Compton scattering, or VCS) has attracted considerable experimental and theoretical interest (for an overview see, e.g., [1–3] and references therein). Both the near-threshold [4] and high-momentum transfer regimes [5–7] of VCS turn out to be very informative for studying the structure of hadrons.

In the below-threshold region a set of structure functions, the so-called generalized polarizabilities (GPs), has been introduced in Ref. [4] in order to parametrize structure-dependent effects in the VCS amplitude to leading order in the final-photon momentum q'. These GPs contain model-dependent information beyond the low-energy theorem (LET) of real [8,9] and virtual Compton scattering [10]. The first experimental results for generalized polarizabilities of the proton have recently been obtained at the Mainz Microtron (MAMI) for a four-momentum squared of $Q^2 = 0.33$ GeV^2 [11]. Additional experiments aiming at proton polarizabilities are presently carried out at Jefferson Lab [12] and MIT-Bates [13]. A sensitivity study of inelastic high-energy pion-electron scattering on the pion VCS amplitude has been performed as part of the Fermilab E781 SELEX experiment [14,15].

Theoretical predictions of the nucleon GPs have been obtained in different frameworks such as the constituent quark model [4,16–18], the linear σ-model [19], chiral perturbation theory (ChPT) [20,21], the Skyrme model [22], models with effective photon-pion-nucleon Lagrangians [23,24], and dispersion theories [25,26]. Pion and kaon GPs have been discussed at the one-loop level of ChPT [27,28],

In the present paper we are not so much concerned with specific results for GPs as obtained from various models of hadrons. Our investigations are rather directed to an analysis of general properties of GPs as well as their physical interpretation which, in our opinion, was only insufficiently covered previously. We mainly consider kinematical aspects, relegating dynamical aspects of properties of the generalized polarizabilities such as sum rules, dispersion relations, etc. to a future publication. In order to avoid complications related with the spin of the target we only discuss the simplest case of a (pseudo) scalar particle like the pion. Our considerations may as well be applied to the spin-averaged part of the VCS amplitude for a target of arbitrary spin. For illustrative purposes we make use of results obtained in the framework of ChPT.

Our work is organized as follows. In Sec. II, we introduce a new method of obtaining invariant amplitudes of electromagnetic reactions free from kinematical singularities and constraints, using (double) virtual Compton scattering as an example. Section III contains a motivation for choosing a specific (though standard) form of the Born amplitude which is used for separating convectional from internal contributions in the scattering amplitude. In Sec. IV, we show how generalized dipole polarizabilities can be defined in a Lorentz-invariant manner from the invariant amplitudes without introducing inconvenient singular kinematical factors. In particular, we show that it is natural to introduce one more dipole polarizability, namely, the transverse electric one, which is needed in order to recover the electric polarization of the target induced by a soft external electric field. This quantity does not contribute in the $\mathcal{O}(q')$ limit of previous analyses. We provide another explanation why

to this order only two of the three spin-independent polarizabilities are accessible in photon electroproduction experiments. We give a physical interpretation of the generalized dipole polarizabilities in terms of densities of the induced electric and magnetic polarizations. In Sec. V we argue that the associated space distributions are meaningful even for a light particle like the pion. We determine such space distributions at large distances $r \sim 1/m_\pi$ derived from various form factors using the predictions of ChPT. In passing, we also include $\mathrm{SU}(3)_f$ extensions of previous results obtained within two-flavor HBChPT. Analytical results of spatial distributions obtained via dispersion relations are relegated to the Appendix.

II. TENSOR BASIS AND INVARIANT AMPLITUDES OF VCS

In our discussion of virtual Compton scattering off a spinless hadron, we will often refer to this hadron as a “pion,” although our general considerations also hold true for other spinless particles, nuclei, and even atoms, as well as for spin-averaged amplitudes. We start our analysis with an investigation of the general kinematical structure of the pion VCS amplitude T_{VCS} for the case of two virtual photons, assuming that both initial and final pions are on shell. Our aim is to construct a tensor basis and a related set of Lorentz-invariant amplitudes B_i, that provides a complete parametrization of T_{VCS}. We require all B_i to be free from kinematical singularities and constraints, because this simplifies the classification of low-energy characteristics of the pion, and also provides technical advantages, for instance when discussing dispersion relations.

The problem of finding a set of amplitudes for VCS was already addressed and solved by Tarrach for both spin-0 and spin-1/2 targets [29] by using a projection technique originally proposed by Bardeen and Tung [30]. Although, in principle, we could directly use the results of Ref. [29], we prefer to construct the tensor basis again, first, in order to demonstrate a very simple and powerful alternative method which avoids projections and, second, in order to rearrange the VCS tensor in a manifestly gauge-invariant form and to split it into structures contributing to real Compton scattering, VCS with one photon virtual and, finally, VCS with both photons virtual.

To begin with, we define the amplitude T_{VCS} of virtual Compton scattering,

$$\gamma(\epsilon, q) + \pi(p) \to \gamma(\epsilon', q') + \pi(p'), \tag{2.1}$$

as

$$T_{\mathrm{VCS}} = \epsilon_\mu \epsilon'^*_\nu T^{\mu\nu}, \tag{2.2}$$

where $T_{\mu\nu}$ is the Compton tensor given in terms of the covariant (Wick) T_W-product of the electromagnetic currents,

$$T_{\mu\nu} = \int \langle \pi(p') | i T_W [j_\mu(x) j_\nu(0)] | \pi(p) \rangle \, e^{-iq\cdot x} \, d^4x. \tag{2.3}$$

We normalize all single-particle states as

$$\langle \boldsymbol{p}' | \boldsymbol{p} \rangle = 2p_0 (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}'), \tag{2.4}$$

so that in case of real photons the S-matrix reads

$$S_{fi} = i(2\pi)^4 \delta^4(p + q - p' - q')\, T_{\mathrm{VCS}}. \tag{2.5}$$

Due to four-momentum conservation, $p+q = p'+q'$, the tensor $T_{\mu\nu}$ depends on three linearly independent vectors P, Q, and R:

$$P = \frac{1}{2}(p + p'), \quad Q = \frac{1}{2}(q + q'), \quad R = \frac{1}{2}(p' - p) = \frac{1}{2}(q - q'). \tag{2.6}$$

Since we only consider the case of initial and final pions on the mass shell, the vectors P, Q, and R are constrained by

$$P^2 = M^2 - R^2, \quad P \cdot R = 0, \tag{2.7}$$

where M is the pion mass. Hence, we can choose four independent kinematical invariants which, for the moment, we take to be Q^2, R^2, $P \cdot Q$, and $Q \cdot R$.

The discrete symmetries as well as gauge invariance impose restrictions on the general form of the Compton tensor. For charged pions the combination of pion crossing with charge-conjugation symmetry results in[1]

$$T_{\mu\nu}(P, Q, R) = T_{\mu\nu}(-P, Q, R), \tag{2.8}$$

whereas photon crossing yields

$$T_{\mu\nu}(P, Q, R) = T_{\nu\mu}(P, -Q, R). \tag{2.9}$$

Gauge invariance requires

$$\begin{aligned} q_\mu T^{\mu\nu} &= (Q + R)_\mu T^{\mu\nu} = 0, \\ q'_\nu T^{\mu\nu} &= (Q - R)_\nu T^{\mu\nu} = 0, \end{aligned} \tag{2.10}$$

where the first and second equation are not independent once photon-crossing symmetry is imposed. Finding a solution to Eqs. (2.10) without introducing kinematical singularities or constraints is the main challenge in constructing appropriate amplitudes.

[1]For the neutral pion (but not for the neutral kaon), which is its own antiparticle, only pion crossing is required to obtain Eq. (2.8).

Because of parity conservation the Compton tensor transforms as a proper second-rank Lorentz tensor. The most general $T_{\mu\nu}$ satisfying the crossing-symmetry conditions of Eqs. (2.8) and (2.9) can be written as a linear combination

$$T_{\mu\nu} = \sum_{i=1}^{10} \tau^i_{\mu\nu} A_i \tag{2.11}$$

of ten basis tensors $\tau^i_{\mu\nu}$ which include the metric tensor $g_{\mu\nu}$ and 9 bi-linear products of P, K, and Q:[2]

$$\begin{aligned}
\tau^1_{\mu\nu} &= g_{\mu\nu},\\
\tau^2_{\mu\nu} &= P_\mu P_\nu,\\
\tau^3_{\mu\nu} &= Q_\mu Q_\nu,\\
\tau^4_{\mu\nu} &= R_\mu R_\nu,\\
\tau^5_{\mu\nu} &= (P_\mu Q_\nu + P_\nu Q_\mu)(P\cdot Q),\\
\tau^6_{\mu\nu} &= (P_\mu Q_\nu - P_\nu Q_\mu)(P\cdot Q)(Q\cdot R),\\
\tau^7_{\mu\nu} &= (P_\mu R_\nu + P_\nu R_\mu)(P\cdot Q)(Q\cdot R),\\
\tau^8_{\mu\nu} &= (P_\mu R_\nu - P_\nu R_\mu)(P\cdot Q),\\
\tau^9_{\mu\nu} &= (Q_\mu R_\nu + Q_\nu R_\mu)(Q\cdot R),\\
\tau^{10}_{\mu\nu} &= (Q_\mu R_\nu - Q_\nu R_\mu).
\end{aligned} \tag{2.12}$$

Appropriate factors of $P\cdot Q$ have been introduced such that all $\tau^i_{\mu\nu}$ are even functions of P, as required by Eq. (2.8). Correspondingly, factors of $Q\cdot R$ provide for photon-crossing-even basis tensors [see Eq. (2.9)]. With such a choice of $\tau^i_{\mu\nu}$, all the functions A_i depend on the crossing-even variables Q^2, R^2, $(P\cdot Q)^2$, and $(Q\cdot R)^2$.

At this point it might be worthwhile to explain why introducing factors of $P\cdot Q$ and $Q\cdot R$ into Eqs. (2.12) is harmless to the analytical properties of the functions A_i in Eq. (2.11). For that purpose, let us omit in Eqs. (2.12) all factors of $P\cdot Q$ or $Q\cdot R$, and denote the resulting basis tensors by $\mathring{\tau}^i_{\mu\nu}$ with $\mathring{A}_i$ the corresponding functions of the expansion of $T_{\mu\nu}$ in terms of the "reduced" tensors $\mathring{\tau}^i_{\mu\nu}$. Let us further assume that the tensor $T_{\mu\nu}$ results from some set of Feynman diagrams consistent with all symmetries. Consider an arbitrary Feynman diagram denoted by G. Obviously, its contribution to $T_{\mu\nu}$ can be expressed in terms of $g_{\mu\nu}$ or bi-linear products of four-momenta (these are just the $\mathring{\tau}^i_{\mu\nu}$) multiplied by scalar coefficients. Any such coefficient will at most have *dynamical* singularities related with propagators of intermediate particles but no *kinematical* singularities. Stated differently, all individual contributions $\mathring{A}^G_i$ to the functions $\mathring{A}_i$ are free from kinematical singularities. In general, a single contribution is not separately crossing symmetric. Eventually, crossing symmetry of $T_{\mu\nu}$ is obtained after

[2]Due to parity, structures containing a single fully antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ are excluded.

adding one or several crossed partners, G_c, of the diagram G. Let us consider, for instance, the tensor structure $\mathring{\tau}^5_{\mu\nu} = P_\mu Q_\nu + P_\nu Q_\mu$ which is odd under $P \to -P$. We can then write

$$\mathring{A}_5^{G+G_c}(x) = \mathring{A}_5^{G}(x) - \mathring{A}_5^{G}(-x), \tag{2.13}$$

where we introduced $x = P \cdot Q$ and, for brevity, omitted P-independent arguments like Q^2. The second term in Eq. (2.13) represents the contribution of the crossed diagram G_c which makes the amplitude $\mathring{A}_5^{G+G_c}(x)$ odd. Note that both contributions are non-singular as $x \to 0$. From Eq. (2.13) we conclude that $\mathring{A}_5^{G+G_c}(x) = x A_5^{G+G_c}(x)$, where $A_5^{G+G_c}(x)$ is an even function of x which has no pole at $x = 0$ and which therefore has no kinematical singularities at all. In other words, the sum of all Feynman diagrams contains the term $\tau^5_{\mu\nu} A_5$, the tensor $\tau^5_{\mu\nu}$ carrying a factor of $P \cdot Q$ and the function A_5 having no kinematical singularities.

The same consideration immediately applies to the tensors $\tau^8_{\mu\nu}$ and $\tau^9_{\mu\nu}$. In the case of $\tau^6_{\mu\nu}$ or $\tau^7_{\mu\nu}$, we have to apply the procedure given by Eq. (2.13) twice — first for showing that the corresponding function $\mathring{A}_i$ contains the factor of $P \cdot Q$, and second for showing that it contains the factor of $Q \cdot R$ as well. Eventually, we conclude that all the coefficients A_i in Eq. (2.11) are free from kinematical singularities.

By means of the factors $P \cdot Q$ and $Q \cdot R$ in the tensor basis Eq. (2.12) we can solve the constraints of Eqs. (2.10) without introducing singular coefficients and without using the projectors suggested in Ref. [30]. Indeed, inserting Eq. (2.11) for $T^{\mu\nu}$ into Eqs. (2.10) and collecting coefficients of the independent four-momenta P, Q, and R, we obtain a set of six linear equations in the functions A_i, of which only five are independent:

$$\begin{aligned}
&A_2 + Q^2 A_5 - (Q \cdot R)^2 A_6 + (Q \cdot R)^2 A_7 - R^2 A_8 = 0,\\
&A_5 - Q^2 A_6 + R^2 A_7 - A_8 = 0,\\
&A_1 + Q^2 A_3 + (P \cdot Q)^2 A_5 + (Q \cdot R)^2 A_9 - R^2 A_{10} = 0,\\
&A_3 + (P \cdot Q)^2 A_6 + R^2 A_9 - A_{10} = 0,\\
&A_1 + R^2 A_4 + (P \cdot Q)^2 A_8 + (Q \cdot R)^2 A_9 + Q^2 A_{10} = 0,\\
&A_4 + (P \cdot Q)^2 A_7 + Q^2 A_9 + A_{10} = 0.
\end{aligned} \tag{2.14}$$

One can now express A_i, $i = 1, \cdots, 5$, in terms of the remaining functions A_i, $i = 6, \cdots, 10$, without singular coefficients:

$$\begin{aligned}
A_1 &= R^2 (P \cdot Q)^2 A_7 - (P \cdot Q)^2 A_8 + [Q^2 R^2 - (Q \cdot R)^2] A_9 + (R^2 - Q^2) A_{10},\\
A_2 &= [(Q \cdot R)^2 - (Q^2)^2] A_6 + [Q^2 R^2 - (Q \cdot R)^2] A_7 + (R^2 - Q^2) A_8,\\
A_3 &= -(P \cdot Q)^2 A_6 - R^2 A_9 + A_{10},\\
A_4 &= -(P \cdot Q)^2 A_7 - Q^2 A_9 - A_{10},\\
A_5 &= Q^2 A_6 - R^2 A_7 + A_8.
\end{aligned} \tag{2.15}$$

Using Eqs. (2.15), we can finally rewrite Eq. (2.11) as

$$T_{\mu\nu} = \sum_{i=6}^{10} T^i_{\mu\nu} A_i(Q^2, R^2, (P\cdot Q)^2, (Q\cdot R)^2), \tag{2.16}$$

where

$$\begin{aligned}
T^6_{\mu\nu} &= [(Q\cdot R)^2 - (Q^2)^2] P_\mu P_\nu - (P\cdot Q)^2 Q_\mu Q_\nu \\
&\quad + Q^2 (P\cdot Q)(P_\mu Q_\nu + P_\nu Q_\mu) + (P\cdot Q)(Q\cdot R)(P_\mu Q_\nu - P_\nu Q_\mu), \\
T^7_{\mu\nu} &= R^2 (P\cdot Q)^2 g_{\mu\nu} + [Q^2 R^2 - (Q\cdot R)^2] P_\mu P_\nu - (P\cdot Q)^2 R_\mu R_\nu \\
&\quad - R^2 (P\cdot Q)(P_\mu Q_\nu + P_\nu Q_\mu) + (P\cdot Q)(Q\cdot R)(P_\mu R_\nu + P_\nu R_\mu), \\
T^8_{\mu\nu} &= -(P\cdot Q)^2 g_{\mu\nu} + [R^2 - Q^2] P_\mu P_\nu \\
&\quad + (P\cdot Q)(P_\mu Q_\nu + P_\nu Q_\mu) + (P\cdot Q)(P_\mu R_\nu - P_\nu R_\mu), \\
T^9_{\mu\nu} &= [Q^2 R^2 - (Q\cdot R)^2] g_{\mu\nu} - R^2 Q_\mu Q_\nu - Q^2 R_\mu R_\nu + (Q\cdot R)(Q_\mu R_\nu + Q_\nu R_\mu), \\
T^{10}_{\mu\nu} &= (R^2 - Q^2) g_{\mu\nu} + Q_\mu Q_\nu - R_\mu R_\nu + (Q_\mu R_\nu - Q_\nu R_\mu)
\end{aligned} \tag{2.17}$$

are five basis tensors which explicitly satisfy the crossing-symmetry and gauge-invariance conditions of Eqs. (2.8)–(2.10). Respectively, the five functions A_i, $i = 6, \cdots, 10$, can be considered as invariant amplitudes of virtual Compton scattering which are free from kinematical singularities and constraints.

In passing we note that the same method of constructing a basis and invariant amplitudes free from kinematical singularities and constraints also works for the VCS amplitude of a spin-1/2 target such as the nucleon.

The tensors $T^8_{\mu\nu}$ and $T^{10}_{\mu\nu}$ have exactly the same form as for real Compton scattering and can easily be identified with the more common notation

$$\begin{aligned}
T^8_{\mu\nu} &= -(P\cdot q)(P\cdot q') g_{\mu\nu} - (q\cdot q') P_\mu P_\nu + (P\cdot q') P_\mu q_\nu + (P\cdot q) P_\nu q'_\mu, \\
T^{10}_{\mu\nu} &= q'_\mu q_\nu - (q\cdot q') g_{\mu\nu}.
\end{aligned} \tag{2.18}$$

However, also the remaining tensors $T^i_{\mu\nu}$, $i = 6, 7, 9$, and thus the corresponding functions A_i contribute to RCS. This, clearly, is a drawback of the basis of Eqs. (2.17) and it would be convenient to have, instead, a basis such that *exclusively* the two tensors $T^8_{\mu\nu}$ and $T^{10}_{\mu\nu}$ contribute to RCS, another one appears for the case of one virtual photon and, finally, the two remaining structures also contribute to $\gamma^*\pi \to \gamma^*\pi$. To that end let us introduce gauge-invariant combinations of photon polarizations and momenta,

$$F_{\mu\nu} = -i(q_\mu \epsilon_\nu - q_\nu \epsilon_\mu), \quad F'_{\mu\nu} = i(q'_\mu \epsilon'^*_\nu - q'_\nu \epsilon'^*_\mu). \tag{2.19}$$

These second-rank tensors represent the Fourier components of the electromagnetic field-strength tensor $\mathcal{F}_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ associated with plane-wave initial and final photons described by vector potentials $A_\mu(x) = \epsilon_\mu \exp(-iq\cdot x)$ and $A'_\mu(x) = \epsilon'^*_\mu \exp(iq'\cdot x)$, respectively. In terms of $F_{\mu\nu}$ and $F'_{\mu\nu}$ it turns out to be rather straightforward to identify structures contributing for real or virtual photons.

For example, the A_{10} contribution to the VCS amplitude reads

$$\epsilon^\mu \epsilon'^{*\nu} T^{10}_{\mu\nu} A_{10} = -\frac{1}{2} F^{\mu\nu} F'_{\mu\nu} A_{10} \tag{2.20}$$

which can be interpreted as the matrix element of the effective Lagrangian[3]

$$\mathcal{L} = -\frac{1}{4} \mathcal{A}_{10} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \phi^\dagger \phi. \tag{2.21}$$

Here $\mathcal{A}_{10}$ represents a differential operator in terms of (covariant) derivatives acting on both pion and electromagnetic fields with Fourier components given by the function A_{10}. Similarly, the contribution of the amplitude A_8 can be written as

$$\epsilon^\mu \epsilon'^{*\nu} T^{8}_{\mu\nu} A_8 = -(P_\mu F^{\mu\nu})(P^\rho F'_{\rho\nu}) A_8 \tag{2.22}$$

which results from the effective interaction Lagrangian

$$\mathcal{L} = -\frac{1}{2} \mathcal{A}_8 \mathcal{F}^{\alpha\nu} \mathcal{F}_{\beta\nu} \hat{P}_\alpha \hat{P}^\beta \phi^\dagger \phi, \tag{2.23}$$

where the action of $\hat{P}_\alpha \hat{P}_\beta \phi^\dagger \phi$ is defined by

$$\begin{aligned} \hat{P}_\beta \phi^\dagger \phi &= \frac{i}{2} \phi^\dagger D_\beta \phi - \frac{i}{2} (D_\beta \phi)^\dagger \phi, \\ \hat{P}_\alpha \hat{P}_\beta \phi^\dagger \phi &= -\frac{1}{4} \phi^\dagger D_\alpha D_\beta \phi + \frac{1}{4} (D_\alpha \phi)^\dagger D_\beta \phi + \frac{1}{4} (D_\beta \phi)^\dagger D_\alpha \phi - \frac{1}{4} (D_\alpha D_\beta \phi)^\dagger \phi, \end{aligned} \tag{2.24}$$

and $D_\alpha \phi = \partial_\alpha \phi + ieZ A_\alpha \phi$ with $e > 0$, $e^2/4\pi \approx 1/137$, and Ze denoting the charge of the particle.

The tensor structures $T^i_{\mu\nu}$, $i = 6, 7, 9$, involve higher powers of photon momenta and thus are related with derivatives of the electromagnetic fields. Introducing the four-vectors

$$\begin{aligned} -iq^\mu F_{\mu\nu} &= -q^2 \epsilon_\nu + (q \cdot \epsilon) q_\nu, \\ iq'^\mu F'_{\mu\nu} &= -q'^2 \epsilon'^*_\nu + (q' \cdot \epsilon'^*) q'_\nu, \end{aligned} \tag{2.25}$$

which vanish for real photons, we obtain two more tensors, namely $T^9_{\mu\nu}$ and $T^6_{\mu\nu} - T^7_{\mu\nu}$, by using the identities

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[3]Here, we assume that "pion" and "antipion" are different particles, such as π^+ and π^- or K^0 and $\bar{K}^0$. For the case of a charged pion, the field $\phi \equiv \pi^+$ is given in terms of the Hermitian, Cartesian isospin components ϕ_i as $\phi = (\phi_1 - i\phi_2)/\sqrt{2}$ ($\phi^\dagger \equiv \pi^- = (\phi_1 + i\phi_2)/\sqrt{2}$) and destroys a π^+ (π^-). In the case of the neutral pion, we have to take $\phi = \phi^\dagger = \phi_3$ and replace the factor of 1/4 in Eq. (2.21) by 1/8. These trivial changes also apply to other Lagrangians written below.

$$(q_\mu F^{\mu\nu})(q'^\rho F'_{\rho\nu}) = \epsilon^\mu \epsilon'^{*\nu}[4T^9_{\mu\nu} + (R^2 - Q^2)T^{10}_{\mu\nu}] \tag{2.26}$$

and

$$(P_\nu q_\mu F^{\mu\nu})(P^\sigma q'^\rho F'_{\rho\sigma}) = \epsilon^\mu \epsilon'^{*\nu}[-2T^6_{\mu\nu} + 2T^7_{\mu\nu} + (R^2 + Q^2)T^8_{\mu\nu} - (P\cdot Q)^2 T^{10}_{\mu\nu}]. \tag{2.27}$$

The remaining linear combination $T^6_{\mu\nu} + T^7_{\mu\nu}$ is contained in a product similar to Eq. (2.27), however with photon momenta interchanged in one factor:

$$\begin{aligned}&(P_\nu F^{\mu\nu} q'_\mu)(P^\sigma q'^\rho F'_{\rho\sigma}) + (P^\nu F'_{\mu\nu} q^\mu)(P_\sigma q_\rho F^{\rho\sigma})\\ &\quad = \epsilon^\mu \epsilon'^{*\nu}(-2T^6_{\mu\nu} - 2T^7_{\mu\nu} - 2R^2 T^8_{\mu\nu}).\end{aligned} \tag{2.28}$$

With the above identities the most general VCS amplitude can be written in the following manifestly gauge-invariant form:

$$\begin{aligned}T_{\rm VCS} = {}& \frac{1}{2} F^{\mu\nu} F'_{\mu\nu} B_1 + (P_\mu F^{\mu\nu})(P^\rho F'_{\rho\nu}) B_2\\ &+ [(P^\nu q'^\mu F_{\mu\nu})(P^\sigma q'^\rho F'_{\rho\sigma}) + (P^\nu q^\mu F_{\mu\nu})(P^\sigma q^\rho F'_{\rho\sigma})] B_3\\ &+ (q_\mu F^{\mu\nu})(q'^\rho F'_{\rho\nu}) B_4 + (P^\nu q^\mu F_{\mu\nu})(P^\sigma q'^\rho F'_{\rho\sigma}) B_5.\end{aligned} \tag{2.29}$$

Here all the invariant amplitudes B_i are free from kinematical singularities and constraints, because the transformation from the basis $T^i_{\mu\nu}$ of Eq. (2.17) to the basis of Eq. (2.29) is non-singular. This can also be easily seen from the following relations between the two sets of amplitudes A_i and B_i:

$$\begin{aligned}B_1 &= -A_{10} + \frac{(P\cdot Q)^2}{4}(A_6 - A_7) + \frac{R^2 - Q^2}{4} A_9,\\ B_2 &= -A_8 + \frac{R^2}{2}(A_6 + A_7) - \frac{R^2 + Q^2}{4}(A_6 - A_7),\\ B_3 &= -\frac{1}{4}(A_6 + A_7), \qquad B_4 = \frac{1}{4} A_9, \qquad B_5 = \frac{1}{4}(A_7 - A_6).\end{aligned} \tag{2.30}$$

The determinant of the transformation expressing B_i ($i = 1, \cdots, 5$) in terms of A_i ($i = 6, \cdots, 10$) is $1/32 \neq 0$, independently of the values of the kinematical variables.

In view of the rather compact and transparent structure of Eq. (2.29), we will use in the following the parametrization of $T_{\rm VCS}$ given by the amplitudes B_i. As seen from Eqs. (2.25) and (2.29), only the amplitudes B_1 and B_2 are needed to describe real Compton scattering, because then $q^\mu F_{\mu\nu} = q'^\mu F'_{\mu\nu} = 0$. When only one photon is virtual, one more amplitude (B_3) contributes. All five amplitudes B_i enter, when both photons are virtual.

Equation (2.29) can be interpreted as the matrix element of the effective interaction

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{4}[\hat{B}_1 (\mathcal{F}_{\mu\nu})^2 + 2\hat{B}_4 (\partial^\mu \mathcal{F}_{\mu\nu})^2]\phi^\dagger\phi\\ &+ \frac{1}{2}[\hat{B}_2 \mathcal{F}^{\alpha\mu}\mathcal{F}_{\beta\mu} + \hat{B}_5 (\partial_\mu \mathcal{F}^{\alpha\mu})(\partial^\nu \mathcal{F}_{\beta\nu}) - 2\hat{B}_3 \mathcal{F}^{\alpha\mu}(\partial_\mu \partial^\nu \mathcal{F}_{\beta\nu})]\hat{P}_\alpha \hat{P}^\beta \phi^\dagger\phi,\end{aligned} \tag{2.31}$$

where $\hat{B}_i$ are differential operators acting on all the fields and determined by their Fourier components B_i.

Of course, after substituting $q \to q_1$, $q' \to -q_2$, $p \to -p_1$, and $p' \to p_2$, Eq. (2.29) also describes the general kinematical structure of the amplitude of the crossed reaction $\gamma(q_1)\gamma(q_2) \to \pi(p_1)\pi(p_2)$ for on-shell pions. Exactly the same considerations apply to any other spin-0 hadron assuming the same symmetry principles (Lorentz and gauge invariance, P, T, and C conservation) and are also applicable to a properly spin-averaged VCS amplitude for hadrons with finite spin.

As mentioned before, the functions B_i depend on the four invariants Q^2, R^2, $(Q \cdot R)^2$ and $(P \cdot Q)^2$. As an alternative, the following combinations of the first 3 quantities can be used as independent arguments of B_i: $q^2 + q'^2 = 2(R^2 + Q^2)$, $q \cdot q' = R^2 - Q^2$, and $q^2 q'^2 = (R^2 + Q^2)^2 - 4(Q \cdot R)^2$. Thus, we may write

$$B_i = B_i(\nu^2, q \cdot q', q^2 + q'^2, q^2 q'^2), \tag{2.32}$$

where ν is defined as

$$M\nu = P \cdot Q = P \cdot q = P \cdot q'. \tag{2.33}$$

Besides being manifestly crossing symmetric, this form of B_i has the advantage of having a simple limit if one or both photons become real.

Finally, the Mandelstam invariants of the VCS reaction read

$$\begin{aligned} s &= (p+q)^2 = M^2 + 2M\nu + q \cdot q', \\ u &= (p-q')^2 = M^2 - 2M\nu + q \cdot q', \\ t &= (q-q')^2 = q^2 + q'^2 - 2q \cdot q'. \end{aligned} \tag{2.34}$$

III. THE BORN TERMS AND GAUGE INVARIANCE

In order to describe the internal structure of the pion in terms of its generalized polarizabilities as tested in virtual Compton scattering, we first have to isolate a convection contribution which originates in two successive interactions of the photons with the electromagnetic current of the pion, resulting in singularities at zero photon momenta. For a point-like (pseudo-) scalar particle of electric charge eZ, the interaction with an external electromagnetic field can be described in terms of the Lagrangian[4]

[4]Since we do not treat the electromagnetic field as a dynamical variable, it will not be included in the list of arguments of the Lagrangian.

$$\mathcal{L}_0(D_\mu\phi,\phi) = D_\mu\phi(D^\mu\phi)^\dagger - M^2\phi\phi^\dagger, \tag{3.1}$$

where the covariant derivative

$$D_\mu\phi = (\partial_\mu + ieZA_\mu)\phi \tag{3.2}$$

ensures the invariance of the Lagrangian under the *canonical* gauge transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\Lambda(x), \qquad \phi(x) \mapsto \exp[-ieZ\Lambda(x)]\phi(x). \tag{3.3}$$

However, such a description is not sufficient for an extended particle and we have to modify the above effective Lagrangian.

To that end, let us first consider a classical system of n constituents with charges $e_a = eZ_a$ and masses m_a which is exposed to a static external potential $A_0(\boldsymbol{r})$. The electrostatic energy of such a system is given by

$$W = \sum_{a=1}^{n} e_a A_0(\boldsymbol{r}_a) = \sum_{a=1}^{n} e_a \Big[A_0(\boldsymbol{R}) + \rho_{ai}\nabla_i A_0(\boldsymbol{R}) + \frac{1}{2}\rho_{ai}\rho_{aj}\nabla_i\nabla_j A_0(\boldsymbol{R}) + \cdots\Big], \tag{3.4}$$

where $\boldsymbol{R}$ is the center of mass of the charge distribution and $\boldsymbol{\rho}_a = \boldsymbol{r}_a - \boldsymbol{R}$ are relative coordinates. In the continuum limit, the expression for a spherically symmetric distribution reads

$$W = eZA_0(\boldsymbol{R}) + \frac{e}{6}\langle Zr_E^2\rangle \boldsymbol{\nabla}^2 A_0(\boldsymbol{R}) + \cdots = eF(\boldsymbol{\nabla}^2)A_0(\boldsymbol{R}), \tag{3.5}$$

where $\sum_a e_a \to \int \rho(r)\, d\boldsymbol{r} = Ze$ is the total charge, $\sum_a e_a \boldsymbol{\rho}_a^2 \to \int r^2\rho(r)\, d\boldsymbol{r} = e\langle Zr_E^2\rangle$ is the electric mean square radius, and $F(-\boldsymbol{q}^2) = Z - \frac{1}{6}\langle Zr_E^2\rangle\boldsymbol{q}^2 + \cdots$ is the electric form factor. Of course, the response of the extended system to the external field is determined by the potential and its derivatives together with the corresponding moments of the charge distribution.

A relativistic generalization to the case of an extended pion suggests the following substitution for the vector potential in Eq. (3.2)

$$ZA_\mu(x) \to F(-\partial^2)A_\mu(x) \tag{3.6}$$

which leads to the effective Lagrangian[5]

$$\mathcal{L}_{\text{eff}}(\partial_\mu\phi,\phi) = [\partial_\mu + ieF(-\partial^2)A_\mu]\phi[\partial^\mu - ieF(-\partial^2)A^\mu]\phi^\dagger - M^2\phi\phi^\dagger, \tag{3.7}$$

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[5]Since we want to apply the Lagrangian for the case of space-like virtual photons, we assume the form factor to be real.

and to the first-order electromagnetic vertex

$$\Gamma_\mu(p',p) = (p'+p)_\mu F(q^2), \quad q = p'-p, \quad F(0) = Z. \tag{3.8}$$

Note that Eq. (3.7) is no longer invariant under the canonical gauge transformation of Eq. (3.3) and, correspondingly, the electromagnetic vertex of Eq. (3.8) does not satisfy the Ward-Fradkin-Takahashi identity [32–34]

$$q_\mu \Gamma^\mu(p',p) \neq Z[\Delta^{-1}(p') - \Delta^{-1}(p)], \tag{3.9}$$

where $\Delta(p) = 1/(p^2 - M^2)$ is the free propagator of $\mathcal{L}_0(\partial_\mu\phi, \phi)$. In fact, a different gauge transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\Lambda(x), \qquad \phi(x) \mapsto \exp[-ieF(-\partial^2)\Lambda(x)]\phi(x), \tag{3.10}$$

defines a local realization of the symmetry group U(1), i.e., under two successive transformations described by smoothly varying functions Λ_1 and Λ_2, the fields transform as

$$\begin{aligned} A_\mu &\mapsto A_\mu + \partial_\mu\Lambda_1 \mapsto (A_\mu + \partial_\mu\Lambda_1) + \partial_\mu\Lambda_2 = A_\mu + \partial_\mu(\Lambda_1 + \Lambda_2), \\ \phi &\mapsto \exp[-ieF(-\partial^2)\Lambda_1]\phi \mapsto \exp[-ieF(-\partial^2)\Lambda_2]\exp[-ieF(-\partial^2)\Lambda_1]\phi \\ &= \exp[-ieF(-\partial^2)(\Lambda_1 + \Lambda_2)]\phi. \end{aligned} \tag{3.11}$$

Accordingly, we define a *noncanonical* covariant derivative as

$$\tilde{D}_\mu\phi = [\partial_\mu + ieF(-\partial^2)A_\mu]\phi, \tag{3.12}$$

such that $\tilde{D}_\mu\phi$ transforms in the same way as ϕ under Eq. (3.10) and the effective Lagrangian

$$\mathcal{L}_{\text{eff}}(\partial_\mu\phi, \phi) = \mathcal{L}_0(\tilde{D}_\mu\phi, \phi) = \tilde{D}_\mu\phi(\tilde{D}^\mu\phi)^\dagger - M^2\phi\phi^\dagger \tag{3.13}$$

remains invariant under Eq. (3.10). We stress that from a formal point of view *any* (real) function represented by a power series would yield a realization of gauge invariance and that the choice of the electromagnetic form factor is entirely motivated on physical grounds through Eqs. (3.5) and (3.6).

Although a description of a finite-size pion based on Eq. (3.7) is mathematically consistent, it turns out to be inconvenient as soon as interactions of several particles with different form factors are considered. For instance, even a simple local interaction of fields ϕ_a ($a = 1, \cdots, n$) of the form

$$\mathcal{L}_{\text{int}}(\phi_a) = g\prod_a \phi_a, \tag{3.14}$$

with a vanishing net charge associated with the product of fields, is no longer automatically gauge invariant as soon as different fields transform with different form factors F_a. Under

the gauge transformation of Eq. (3.10), the interaction Lagrangian, Eq. (3.14), picks up a space-time-dependent phase factor

$$\prod_a \exp[-ieF_a(-\partial^2)\Lambda(x)] \neq 1. \tag{3.15}$$

On the other hand, this would not happen for the transformation law of Eq. (3.3), because

$$\prod_a \exp[-ieZ_a\Lambda(x)] = 1, \tag{3.16}$$

provided the electric charge is conserved at the vertex, i.e., $\sum_a Z_a = 0$.

We therefore redefine the field as follows [35]:

$$\phi(x) = \varphi(x) \exp[ief(-\partial^2)\partial_\mu A^\mu(x)], \tag{3.17}$$

where[6]

$$f(q^2) = \frac{1}{q^2}[F(q^2) - Z] \tag{3.18}$$

generates a new field which transforms canonically under Eq. (3.10), i.e.

$$\begin{aligned}
\varphi &= \exp[-ief(-\partial^2)\partial_\mu A^\mu]\phi && (3.19)\\
&\mapsto \exp[-ief(-\partial^2)(\partial_\mu A^\mu + \partial^2\Lambda)] \exp[-ieF(-\partial^2)\Lambda]\phi \\
&= \exp[-ief(-\partial^2)(\partial_\mu A^\mu + \partial^2\Lambda)] \exp\{-ie[Z - \partial^2 f(-\partial^2)]\Lambda\}\phi \\
&= \exp[-ief(-\partial^2)\partial_\mu A^\mu] \exp(-ieZ\Lambda)\phi = \exp(-ieZ\Lambda)\varphi. && (3.20)
\end{aligned}$$

Rewritten in terms of φ, the Lagrangian (3.13) reads

$$\mathcal{L}_0(D^f_\mu\varphi, \varphi) = D^f_\mu\varphi(D^{f\mu}\varphi)^\dagger - M^2\varphi\varphi^\dagger, \tag{3.21}$$

where the second (noncanonical) covariant derivative $D^f_\mu\varphi$ is defined as

$$D^f_\mu\varphi = [\partial_\mu + ieZA_\mu + ief(-\partial^2)\partial^\nu\mathcal{F}_{\mu\nu}]\varphi, \tag{3.22}$$

and $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This second definition ensures canonical gauge invariance and simultaneously accounts for the finite size of the particle.

Organized in powers of the elementary charge, the Lagrangian can be expressed in terms of the canonical covariant derivative as

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[6]Recall our assumption that F can be expanded in an absolutely convergent series.

$$\mathcal{L}_0(D^f_\mu\varphi,\varphi)=\mathcal{L}_0(D_\mu\varphi,\varphi)+\mathcal{L}_1(D_\mu\varphi,\varphi)+\mathcal{L}_2(\varphi). \tag{3.23}$$

$\mathcal{L}_0$ and

$$\mathcal{L}_1(D_\mu\varphi,\varphi)=-ie[\varphi^\dagger\overleftrightarrow{D}_\mu\varphi]\,f(-\partial^2)\partial_\nu\mathcal{F}^{\mu\nu} \tag{3.24}$$

generate the electromagnetic vertex

$$\Gamma_\mu(p',p)=(p'+p)_\mu F(q^2)-q_\mu(p'^2-p^2)f(q^2),\quad q=p'-p, \tag{3.25}$$

which now satisfies the Ward-Fradkin-Takahashi identity

$$q_\mu\Gamma^\mu(p',p)=Z(p'^2-p^2)=Z[\Delta^{-1}(p')-\Delta^{-1}(p)] \tag{3.26}$$

with the free propagator. The last term

$$\mathcal{L}_2(\varphi)=e^2[f(-\partial^2)\partial_\nu\mathcal{F}^{\mu\nu}][f(-\partial^2)\partial^\rho\mathcal{F}_{\mu\rho}]\varphi\varphi^\dagger \tag{3.27}$$

vanishes, when at least one of the photons is real, thus satisfying $\partial_\nu\mathcal{F}^{\mu\nu}=0$.

In the following we *define* the (generalized) Born terms of the virtual Compton scattering amplitude as the VCS amplitude constructed from either of the Lagrangians of Eqs. (3.7) and (3.21). According to the equivalence theorem of Lagrangian field theory [36,37], the scattering amplitude does not depend on the Lagrangian used as long as all external pions are on shell. To be specific, one obtains

$$T^{\text{Born}}_{\mu\nu}=e^2F(q^2)F(q'^2)\left[2g_{\mu\nu}-\frac{(2p+q)_\mu(2p'+q')_\nu}{(p+q)^2-M^2}-\frac{(2p-q')_\nu(2p'-q)_\mu}{(p-q')^2-M^2}\right]. \tag{3.28}$$

In fact, such a form of the Born amplitude is standard for a discussion of structure-dependent characteristics of the target, and we have shown that this is a very natural generalization of the Born amplitude for a point-like particle to the case of a finite-size particle. As discussed in Ref. [38] in detail, such a generalization incorporates all low-energy singularities of the total VCS amplitude so that the non-Born part of the amplitude can be expanded in powers of small photon momenta giving rise to (generalized) polarizabilities.

IV. LOW-MOMENTUM EXPANSION AND GENERALIZED DIPOLE POLARIZABILITIES

A well-known, general method of obtaining the low-energy expansion of a reaction amplitude consists of expanding invariant amplitudes free from kinematical singularities and constraints in a power series (see e.g. [30,39]). Following this method, we first decompose

the invariant amplitudes $B_i(\nu^2, q\cdot q', q^2+q'^2, q^2q'^2)$ into generalized Born and non-Born contributions,

$$B_i = B_i^{\rm Born} + B_i^{\rm NB}, \quad i=1,\cdots,5. \tag{4.1}$$

The generalized Born, or convection contribution is associated with a set of diagrams describing single-pion exchanges in s- and u-channels with $\gamma\pi\pi$ vertices taken in the on-shell regime. As we have seen in the previous section, additional non-pole terms are necessary to render the generalized Born terms gauge invariant. The thus constructed amplitude possesses all the symmetries of the total amplitude $T_{\rm VCS}$ and contains all singularities of $T_{\rm VCS}$ at low energies. Using the specific form of the Born amplitude $T_{\rm VCS}^{\rm Born}$ given in the previous section, we find the (generalized) Born parts of the invariant amplitudes B_i,

$$B_1^{\rm Born} = (q\cdot q')C, \quad B_2^{\rm Born} = -4C, \quad C = \frac{2e^2F(q^2)F(q'^2)}{(s-M^2)(u-M^2)}, \tag{4.2}$$

and $B_i^{\rm Born}=0$ for $i=3,4,5$.

At energies below inelastic thresholds, the non-Born parts of B_i are regular functions of the kinematical variables. They determine the deviation of $T_{\rm VCS}$ from its Born value of Eq. (3.28). In particular, when the momenta of both photons are small, $q\sim q'\to 0$, one obtains

$$T_{\rm VCS} = T_{\rm VCS}^{\rm Born} + \frac{1}{2}F^{\mu\nu}F'_{\mu\nu}b_1(0) + (P_\mu F^{\mu\nu})(P^\rho F'_{\rho\nu})b_2(0) + \mathcal{O}(q^4), \tag{4.3}$$

where the constants $b_i(0)\equiv B_i^{\rm NB}(0,0,0,0)$, $i=1,2$, can be related to the ordinary electric and magnetic dipole polarizabilities of low-energy real Compton scattering,

$$8\pi M\bar\alpha = -b_1(0) - M^2 b_2(0), \quad 8\pi M\bar\beta = b_1(0). \tag{4.4}$$

Equations (4.3) and (4.4) provide a Lorentz-invariant form of the low-energy theorem for virtual Compton scattering up to second order in the photon momenta (for the case of the nucleon, see Ref. [10]).

Now, following the original idea of Guichon *et al.* [4], we consider the case when the final photon is real and has a very small momentum $q'\to 0$, whereas the initial photon momentum q is allowed to be virtual and is not necessarily small. As may be seen from Eq. (2.29), the amplitude of $\gamma^*\pi\to\gamma\pi$ with a real final photon is determined by three invariant amplitudes B_1, B_2, and B_3,[7]

$$T_{\rm VCS} = \frac{1}{2}F^{\mu\nu}F'_{\mu\nu}B_1 + (P_\mu F^{\mu\nu})(P^\rho F'_{\rho\nu})B_2 + (P^\nu q^\mu F_{\mu\nu})(P^\sigma q^\rho F'_{\rho\sigma})B_3. \tag{4.5}$$

[7]The definition of P of Eq. (2.6) differs by a factor of 1/2 from Ref. [27].

This equation has a particularly simple form in the pion Breit frame (PBF) defined by $\boldsymbol{P} = 0$, i.e., $\boldsymbol{p} = -\boldsymbol{p}'$, in which the initial and final pion are treated on a symmetrical footing. Introducing the Fourier components of the electric and magnetic fields as

$$\begin{aligned} \boldsymbol{E} &= i(q_0\boldsymbol{\epsilon} - \boldsymbol{q}\epsilon_0), \quad \boldsymbol{B} = i\boldsymbol{q}\times\boldsymbol{\epsilon}, \\ \boldsymbol{E}' &= -i(q_0'\boldsymbol{\epsilon}'^* - \boldsymbol{q}'\epsilon_0'^*), \quad \boldsymbol{B}' = -i\boldsymbol{q}'\times\boldsymbol{\epsilon}'^*, \end{aligned} \tag{4.6}$$

we can rewrite Eq. (4.5) in the Breit frame as

$$T_{\mathrm{VCS}} = \Big[(\boldsymbol{B}\cdot\boldsymbol{B}')B_1 - (\boldsymbol{E}\cdot\boldsymbol{E}')(B_1 + P^2B_2) + (\boldsymbol{E}\cdot\boldsymbol{q})(\boldsymbol{E}'\cdot\boldsymbol{q})P^2B_3\Big]_{PBF}. \tag{4.7}$$

In general, even after subtraction of the singular (Born) parts of the amplitudes, all B_i^{NB} still depend on the photon energies and the scattering angle and thus describe a series of multipoles and dispersion effects. However, in the limit of $q' \to 0$, only scalar structure functions depending on q^2 survive:

$$b_i(q^2) = B_i^{\mathrm{NB}}(0,0,q^2,0). \tag{4.8}$$

These functions yield the non-Born parts of the coefficients in (4.7) which we interpret as generalized electric and magnetic dipole polarizabilities:

$$\begin{aligned} 8\pi M\beta(q^2) &= b_1(q^2), \\ 8\pi M\alpha_T(q^2) &= -b_1(q^2) - \left(M^2 - \frac{q^2}{4}\right) b_2(q^2), \\ 8\pi M\alpha_L(q^2) &= -b_1(q^2) - \left(M^2 - \frac{q^2}{4}\right) [b_2(q^2) + q^2 b_3(q^2)], \end{aligned} \tag{4.9}$$

where P^2 has been taken in the limit $q' = 0$ as well, i.e., $P^2 = M^2 - q^2/4$. If the initial virtual photon has a transverse polarization in the Breit frame ($\boldsymbol{E} = \boldsymbol{E}_T \perp \boldsymbol{q}$), the pion response to the transverse electric and magnetic fields is determined by the (generalized) transverse electric and magnetic polarizabilities:

$$(T^{\mathrm{NB}}_{\mathrm{VCS}})_T = 8\pi M[(\boldsymbol{E}_T\cdot\boldsymbol{E}')\alpha_T(q^2) + (\boldsymbol{B}\cdot\boldsymbol{B}')\beta(q^2)] + (\text{higher orders in } q'). \tag{4.10}$$

For a longitudinal polarization ($\boldsymbol{E} = \boldsymbol{E}_L \parallel \boldsymbol{q}$) in the Breit frame,

$$(T^{\mathrm{NB}}_{\mathrm{VCS}})_L = 8\pi M(\boldsymbol{E}_L\cdot\boldsymbol{E}')\alpha_L(q^2) + \mathcal{O}(q'^2). \tag{4.11}$$

In the real-photon limit, $q^2 \to 0$, we have

$$\beta(0) = \bar\beta, \quad \alpha_L(0) = \alpha_T(0) = \bar\alpha. \tag{4.12}$$

All thus defined polarizabilities are functions of q^2 free from kinematical singularities. In particular, for pion, kaon, or nucleon targets they are regular functions below the two-pion threshold, $q^2 < 4m_\pi^2$, and this region includes all space-like momenta.

We will now interpret $\alpha_L(q^2)$, $\alpha_T(q^2)$, and $\beta(q^2)$ by means of a semi-classical qualitative picture. To that end, let us consider a system of n polarizable (neutral) constituents at positions $\boldsymbol{r}_a$ $(a = 1, \cdots, n)$ with electric polarizabilities α^a_{ij}. We allow the constituents to be anisotropic, i.e., the polarizability tensors are symmetric, $\alpha^a_{ij} = \alpha^a_{ji}$, but not necessarily diagonal ($\propto \delta_{ij}$). The system will respond to an external electric field $\boldsymbol{E}(t, \boldsymbol{r})$ by acquiring a dipole moment[8]

$$D_i(t) = 4\pi \sum_{a=1}^{n} \alpha^a_{ij}\, E_j(t, \boldsymbol{r}_a). \tag{4.13}$$

Slow oscillations of this dipole moment generate radiation of an outgoing long-wavelength electromagnetic wave $\boldsymbol{E}'(t, \boldsymbol{r})$ through the interaction $-\boldsymbol{D} \cdot \boldsymbol{E}$. For an incoming plane wave with momentum $\boldsymbol{q}$, i.e., $\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E} \exp(i\boldsymbol{q} \cdot \boldsymbol{r})$, the amplitude for a transition to an outgoing plane wave with a very small momentum (viz. $q'\rho \ll 1$, where ρ characterizes the system's extension) reads

$$f_{fi} = 4\pi \sum_{a=1}^{n} \alpha^a_{ij}\, \exp(i\boldsymbol{q} \cdot \boldsymbol{\rho}_a)\, E_i E'_j\,, \tag{4.14}$$

where $\boldsymbol{\rho}_a = \boldsymbol{r}_a - \boldsymbol{R}$ are the positions of the constituents with respect to the center of mass $\boldsymbol{R}$ of the system. The continuum limit of a system with spherical symmetry must be of the form

$$\sum_{a=1}^{n} \alpha^a_{ij}\, \exp(i\boldsymbol{q} \cdot \boldsymbol{\rho}_a) \rightarrow \alpha_{ij}(\boldsymbol{q}) = \alpha_L(q)\hat{q}_i\hat{q}_j + \alpha_T(q)(\delta_{ij} - \hat{q}_i\hat{q}_j), \tag{4.15}$$

where α_L and α_T do not depend on the direction $\hat{q}$ of $\boldsymbol{q}$. In this way we recover the structure of the VCS amplitude given by Eqs. (4.10) and (4.11).

If the system under consideration is exposed to a static and uniform external electric field $\boldsymbol{E}'$, an electric polarization $\boldsymbol{\mathcal{P}}$ is generated which is related to the *density* of the induced electric dipole moments:

$$\mathcal{P}_i(\boldsymbol{r}) = 4\pi \sum_{a=1}^{n} \alpha^a_{ij}\, \delta^3(\boldsymbol{r} - \boldsymbol{r}_a)\, E'_j \equiv 4\pi\alpha_{ij}(\boldsymbol{r} - \boldsymbol{R})\, E'_j\,. \tag{4.16}$$

The tensor $\alpha_{ij}(\boldsymbol{r})$ is nothing else but the Fourier transform of the polarizability tensor of Eq. (4.15):[9]

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[8] The factor of 4π is related with the (standard) use of Gaussian units for the polarizabilities but natural units for charges and fields.

[9] We do not use different symbols for a function $f(t)$ and its Fourier transform $\tilde{f}(\omega)$.

$$\alpha_{ij}(\boldsymbol{r}) = \int \alpha_{ij}(\boldsymbol{q}) \exp(-i\boldsymbol{q}\cdot\boldsymbol{r}) \frac{d\boldsymbol{q}}{(2\pi)^3}. \tag{4.17}$$

If we define

$$B(q) = \frac{\alpha_L(q) - \alpha_T(q)}{q^2}, \tag{4.18}$$

Eq. (4.15) can be rewritten as

$$\alpha_{ij}(\boldsymbol{q}) = \alpha_T(q)\delta_{ij} + B(q)q_i q_j \tag{4.19}$$

such that the Fourier transformation results in

$$\alpha_{ij}(\boldsymbol{r}) = \alpha_T(r)\delta_{ij} - \nabla_i\nabla_j B(r). \tag{4.20}$$

Because of $B = B(r)$, the second term of Eq. (4.20) reads

$$-\nabla_i\nabla_j B(r) = -\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\frac{B'(r)}{r} - \frac{r_i r_j}{r^2}B''(r). \tag{4.21}$$

On the other hand, from Eq. (4.18) written in the form $q^2 B(q) = \alpha_L(q) - \alpha_T(q)$, one obtains for the Fourier transform

$$-\nabla^2 B(r) = -B''(r) - \frac{2}{r}B'(r) = \alpha_L(r) - \alpha_T(r). \tag{4.22}$$

Eliminating $B''(r)$ from Eq. (4.21) allows one to rewrite Eq. (4.20) as

$$\alpha_{ij}(\boldsymbol{r}) = \alpha_L(r)\frac{r_i r_j}{r^2} + \alpha_T(r)\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right) + \frac{3r_i r_j - r^2\delta_{ij}}{r^3}B'(r). \tag{4.23}$$

Finally, the last term of Eq. (4.23) is determined by reexpressing Eq. (4.22) as

$$\frac{d}{dr}[r^2 B'(r)] = -r^2[\alpha_L(r) - \alpha_T(r)], \tag{4.24}$$

which, assuming the boundary condition $\lim_{r\to\infty} r^2 B'(r) = 0$, is solved by[10]

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[10]Instead of Eq. (4.25) we could also use

$$r^2 B'(r) = -\int_0^r r'^2[\alpha_L(r') - \alpha_T(r')]\, dr',$$

resulting from the boundary condition $\lim_{r\to 0} r^2 B'(r) = 0$. Both results are identical, because

$$\int_0^\infty 4\pi r^2[\alpha_L(r) - \alpha_T(r)]\, dr \equiv \alpha_L(q=0) - \alpha_T(q=0) = 0,$$

where we made use of Eq. (4.12).

$$r^2 B'(r) = \int_r^\infty r'^2 [\alpha_L(r') - \alpha_T(r')]\, dr'. \tag{4.25}$$

In other words, given the Fourier transforms

$$\alpha_{L,T}(r) \equiv \int \alpha_{L,T}(q) \exp(-i\boldsymbol{q}\cdot\boldsymbol{r})\, \frac{d\boldsymbol{q}}{(2\pi)^3}, \tag{4.26}$$

the density of the full electric polarizability of the system, $\alpha_{ij} = \sum_a \alpha^a_{ij}$, can be reconstructed as

$$\alpha_{ij}(\boldsymbol{r}) = \alpha_L(r)\hat{r}_i\hat{r}_j + \alpha_T(r)(\delta_{ij} - \hat{r}_i\hat{r}_j) + \frac{3\hat{r}_i\hat{r}_j - \delta_{ij}}{r^3} \int_r^\infty [\alpha_L(r') - \alpha_T(r')]\, r'^2\, dr'. \tag{4.27}$$

In this context, it is important to realize that both longitudinal and transverse polarizabilities, α_L and α_T, respectively, are needed to fully recover the electric polarization $\boldsymbol{\mathcal{P}}$. The longitudinal polarizability is special, though, because it completely specifies the induced polarization charge density of the system,

$$\delta\rho(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{P}}(\boldsymbol{r}) = -4\pi(\boldsymbol{E}'\cdot\boldsymbol{\nabla})\alpha_L(r), \tag{4.28}$$

where we made use of

$$\nabla_i \alpha_{ij}(\boldsymbol{r}) = \nabla_j \alpha_L(r) \tag{4.29}$$

which follows from Eqs. (4.15) and (4.17). Combining partial integration with the divergence theorem, one finds for the Fourier transform of the induced polarization charge

$$\delta\rho(\boldsymbol{q}) \equiv \int \delta\rho(\boldsymbol{r}) \exp(i\boldsymbol{q}\cdot\boldsymbol{r})\, d\boldsymbol{r} = i\boldsymbol{q}\cdot\boldsymbol{\mathcal{P}}(\boldsymbol{q}) = 4\pi i \alpha_L(q)\, \boldsymbol{q}\cdot\boldsymbol{E}'. \tag{4.30}$$

Such an induced charge density is the source of a longitudinal electric (Coulomb) field and thus generates an effective coupling of the type $\boldsymbol{E}_L\cdot\boldsymbol{E}'$.

At the same time, the transverse polarizability α_T describes rotational displacements of charges which do not contribute to $\delta\rho(\boldsymbol{r})$. They can generate electric fields only for a finite frequency, $q_0' \neq 0$. Therefore, in the limit $q_0' \to 0$ the effective coupling $\boldsymbol{E}_T\cdot\boldsymbol{E}'$ should vanish faster than $\boldsymbol{E}_L\cdot\boldsymbol{E}'$. This is indeed the case, as we will see below.

The relation (4.30) suggests an intimate connection between the longitudinal polarizability α_L and the charge-density operator of the system. In a forthcoming publication we will verify this in a quantum-mechanical and fully relativistic framework.

Similar considerations apply to the magnetic part of the VCS amplitude. In this case the electric polarizabilities α^a_{ij} should be replaced by the magnetic ones, β^a_{ij}. Since the magnetic induction is always transverse (i.e., $\boldsymbol{B}\cdot\boldsymbol{q} = 0$), terms containing $\hat{q}_i\hat{q}_j$ in the magnetic analogue

of Eq. (4.15) do not enter any observable and can thus be omitted. Hence, the unobservable "longitudinal" magnetic polarizability $\beta_L(q)$ can for all q be chosen to be identical with the transverse one, $\beta_T(q)$ rather than only at the point $q = 0$, where the equality $\beta_L(0) = \beta_T(0)$ is dictated by analyticity. With this choice $\beta_{ij}(\boldsymbol{q}) = \beta(q)\,\delta_{ij}$, and the analogue of Eq. (4.27) reads

$$\beta_{ij}(\boldsymbol{r}) = \beta(r)\delta_{ij}. \tag{4.31}$$

Then the magnetization $\boldsymbol{\mathcal{M}}$ induced by the uniform external magnetic field and the corresponding induced electric current $\delta\boldsymbol{j}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{\mathcal{M}}(\boldsymbol{r})$ are, up to an arbitrary gradient, given in terms of the density of the magnetic polarizability as

$$\boldsymbol{\mathcal{M}}(\boldsymbol{r}) = 4\pi\beta(\boldsymbol{r}-\boldsymbol{R})\,\boldsymbol{B}', \tag{4.32}$$

where the Fourier transform of $\beta(r)$ is nothing but the generalized magnetic polarizability $\beta(q)$:

$$\beta(r) = \int \beta(q)\exp(-i\boldsymbol{q}\cdot\boldsymbol{r})\,\frac{d\boldsymbol{q}}{(2\pi)^3}. \tag{4.33}$$

Let us conclude this section by recalling that Eq. (4.10) was obtained by keeping the lowest multipolarities of the final (soft) photon. This, however, leads to different powers of the photon momentum q' in such an expansion. For $q' \to 0$, the energy of the initial photon in the Breit frame vanishes as well, because $q_0 = q_0'$ in that frame. It then follows from Eq. (4.6) that the transverse electric field $\boldsymbol{E}_T$ is of higher order in q' than the (transverse) magnetic field $\boldsymbol{B}$. This is also clear from the relation $\boldsymbol{q}^2\boldsymbol{E}_T = -q_0\boldsymbol{q}\times\boldsymbol{B}$. When only terms up to order $\mathcal{O}(q')$ are retained, the transverse electric field does not contribute! In order to translate Eq. (4.7) into a power expansion of the non-Born part of the VCS amplitude up to $\mathcal{O}(q'^2)$, one has to add two more terms proportional to $(q\cdot q')[\partial B_1^{\mathrm{NB}}/\partial(q\cdot q')]_{q'=0}$ and to a similar derivative of the function $B_1 + P^2 B_2$. These terms introduce an additional angular dependence and therefore higher multipoles (quadrupoles).

The physical process of photon electroproduction $e(k) + h \to e'(k') + h' + \gamma(q')$ consists of the Bethe-Heitler contribution, in which the real final photon is emitted by the initial and final electrons, respectively, and the VCS contribution. The virtual photon of the VCS part, $\gamma(q)$, is described in terms of the polarization vector

$$\epsilon_\mu = \frac{e}{q^2}\bar{u}_e(k')\gamma_\mu u_e(k), \quad q = k - k', \tag{4.34}$$

which is determined by the electron-scattering kinematics. For $q' \to 0$, such an ϵ_μ remains finite. Therefore, the transverse electric field $\boldsymbol{E}_T = iq_0\boldsymbol{\epsilon}_T$ created by the electron transition current is of order $\mathcal{O}(q')$ in the Breit frame and is suppressed in comparison with the magnetic and longitudinal electric fields generated by the current. As an immediate consequence the non-Born part of the VCS amplitude to order $\mathcal{O}(q')$ is characterized by *two* structure

functions [viz. $\alpha_L(q^2)$ and $\beta(q^2)$] rather than by all three functions appearing in the dipole approximation.

Although we arrived at this conclusion by considering the VCS amplitude in the Breit frame [$\boldsymbol{p} = (\boldsymbol{q}' - \boldsymbol{q})/2$], it is clear that two independent structure functions characterize the amplitude to order $\mathcal{O}(q')$ in any other frame such as, for example, the center-of-mass (c.m.) frame. This is true because the amplitude itself is Lorentz invariant and at the same time the real-photon energy, $\omega' = |\vec{q}\,'|$, remains of the same order $\mathcal{O}(q')$ for any *finite* Lorentz boost.

The above consideration gives a transparent explanation of the theorem established in Ref. [31], which states that there are only two independent structure functions which determine $\mathcal{O}(q')$ terms in the so-called full amplitude $T^{\rm NB}_{\rm FVCS}$ of virtual Compton scattering for a spin-0 particle in the c.m. frame. Here, "full" refers to the fact that the polarization and the intensity of the initial photon are given by Eq. (4.34).

In terms of the notation introduced by Guichon *et al.* [4], this theorem establishes a linear relation between the c.m. generalized polarizabilities $P^{(01,01)0}$, $P^{(11,11)0}$, and $\hat{P}^{(01,1)0}$, leaving only two of them independent. In the c.m. frame, the transverse electric field is not vanishing, and the above theorem can be re-phrased as establishing a linear combination of the $\boldsymbol{E}_L$, $\boldsymbol{E}_T$ and $\boldsymbol{B}$ responses in the c.m. frame which vanishes when $q' = 0$. This is just the $\boldsymbol{E}_T$ response in the Breit frame.

The explicit relations between the c.m. polarizabilities and the quantities $\alpha_L(q^2)$ and $\beta(q^2)$ appearing at order $\mathcal{O}(q')$ read

$$\begin{aligned} \alpha_L(q^2) &= -\frac{e^2}{4\pi}\sqrt{\frac{3E_{\rm cm}}{(2J+1)M}}\; P^{(01,01)0}(q_{\rm cm}), \\ \beta(q^2) &= -\frac{e^2}{8\pi}\sqrt{\frac{3E_{\rm cm}}{(2J+1)M}}\; P^{(11,11)0}(q_{\rm cm}), \end{aligned} \tag{4.35}$$

where $E_{\rm cm} = M - q^2/(2M)$ and $q_{\rm cm} = \sqrt{-q^2 + q^4/(4M^2)}$ denote the energy and the absolute value of the three-momentum of the initial pion in the c.m. frame at threshold ($q' = 0$). The spin factor $2J+1$ removes a related spin dependence hidden in the quantities $P^{(\rho' L',\rho L)S}$ and is needed when our "pion" represents a spin-averaged hadron of spin $J \neq 0$.

When considering the Fourier transforms, the additional factor of $\sqrt{E_{\rm cm}}$ in Eq. (4.35) and the use of the Breit-frame momentum transfer $q_{\rm Breit} = \sqrt{-q^2}$ instead of the c.m. momentum transfer $q_{\rm cm}$ will generate a difference for the spatial distributions, especially for such a light particle as the pion. From the analogy with the well-known case of electromagnetic form factors, where spatial distributions are obtained using the Breit-frame variables, we expect that a meaningful Fourier transformation in the case of the generalized polarizabilities also requires the Breit frame. Indeed, the only difference between the kinematics of the reaction of VCS, $\gamma^*\pi \to \gamma'\pi$, and the kinematics of the reaction $\gamma^*\pi \to \pi$, in which the form factors are studied, originates in the presence of an additional photon γ' which carries a vanishing momentum $q' = 0$.

In analogy to Eq. (2.29), the structure-dependent effects as seen in VCS with one (soft or hard) space-like virtual and one soft real photon can be encoded in the following effective Lagrangian:

$$\mathcal{L}_{\text{polariz}} = \frac{1}{4}\hat{b}_1 \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\phi^\dagger\phi + \frac{1}{2}[\hat{b}_2\mathcal{F}^{\alpha\mu}\mathcal{F}_{\beta\mu} - 2\hat{b}_3\mathcal{F}^{\alpha\mu}\,\partial_\mu\partial^\nu\mathcal{F}_{\beta\nu}]\hat{P}_\alpha\hat{P}^\beta\phi^\dagger\phi, \tag{4.36}$$

where $\hat{b}_i$ are differential operators acting on the electromagnetic fields which are determined by their Fourier components $b_i(q^2)$ [see Eq. (4.9)]. The above Lagrangian contains all possible gauge-invariant terms involving at least one field strength tensor $\mathcal{F}_{\mu\nu}$ without derivatives $[\mathcal{O}(\omega')!]$.

V. SPATIAL DISTRIBUTIONS IN CHIRAL PERTURBATION THEORY

A. Preliminary remarks

In accordance with the ideas presented in the previous section, we now consider the Fourier transforms of the q-dependent polarizabilities, generically denoted by $F(q^2)$, and discuss the corresponding spatial distributions $F(r)$.

There is a well-known objection against a straightforward interpretation of such $F(r)$ as a spatial distribution. The argument is related to the fact that the velocities of the target before and after the interaction with the virtual photon depend on the photon momentum. If we think of the target as a composite system of, say, quarks we would expect that the matrix element

$$\int \psi_f^\dagger(\boldsymbol{R}', \tau')\langle \boldsymbol{R}', \tau'|\mathcal{O}|\boldsymbol{R}, \tau\rangle\psi_i(\boldsymbol{R}, \tau)\, d\boldsymbol{R}\, d\tau\, d\boldsymbol{R}'\, d\tau' \tag{5.1}$$

of a transition operator like $\mathcal{O} = \int j^\mu(x)A_\mu(x)\,d^3x$ for a one-photon reaction[11] depends on both the internal (Lorentz-invariant) variables τ of the pion *and* on the pion's center-of-mass variable $\boldsymbol{R}$ (cf. Ref. [40]). Since a relativistic wave function $\psi(\boldsymbol{R}, \tau)$, in general, does not factorize into a product of the type $\phi(\tau)\exp(i\boldsymbol{p}\cdot\boldsymbol{R})$, where $\phi(\tau)$ denotes a $\boldsymbol{p}$-independent internal wave function, some part of the full q-dependence of the transition matrix element may be related with the c.m., partly kinematical effects of $\boldsymbol{p}$ on $\phi(\tau)$. We come closest to associating the Fourier transform $F(r)$ of the *full* matrix element (5.1) with the *internal* spatial structure of the particle by evaluating this matrix element in the Breit frame, in which the pion is at rest on the average, i.e., $\boldsymbol{p} + \boldsymbol{p}' = 0$.

[11] Another example would be $\mathcal{O} = \iint j^\nu(x')A'_\nu(x')G_E(x', x)j^\mu(x)A_\mu(x)\,d^3x\,d^3x'$ involving two currents, where G_E is the Green function.

There is a simple phenomenological argument suggesting that the generalized magnetic polarizability $\beta(q^2)$ defined according to Eq. (4.9) is indeed only related to the internal structure. The point is that this quantity is a function of q^2 without kinematical singularities. In other words, any irregularity in its q^2 behavior is not caused by a Lorentz contraction and has nothing to do with singular γ factors or with quantities like $P_0 = \frac{1}{2}\sqrt{4M^2 + \boldsymbol{q}^2}$. Moreover, even singularity-free quantities like $P_0^2 = P^2$ should be irrelevant, because the momentum scale on which the amplitude $b_1(q^2)$ changes has nothing to do with the particle mass M itself and is fully determined by interactions.

For the other polarizabilities the situation may be more complicated. The sums $\alpha_L(q^2)+\beta(q^2)$ and $\alpha_T(q^2)+\beta(q^2)$ contain an overall factor of P^2 [see Eqs. (4.9)] which, for the pion, introduces a small kinematical mass scale into the q^2 behavior of these sums. So, perhaps a more meaningful consideration of spatial distributions relating to $\alpha_L(r)+\beta(r)$ and $\alpha_T(r)+\beta(r)$ is obtained with the factor P^2/M^2 excluded from $\alpha_L(q^2)+\beta(q^2)$ and $\alpha_T(q^2)+\beta(q^2)$ before performing the Fourier transformations. In the following discussion we will not encounter this problem, because in the theory considered, namely, ChPT at lowest nontrivial order, either the sums $\alpha_L(q^2)+\beta(q^2)$ and $\alpha_T(q^2)+\beta(q^2)$ are exactly zero [for pions and kaons at $\mathcal{O}(p^4)$] or the particle mass M is considered to be infinite [for baryons in HBChPT at $\mathcal{O}(p^3)$].

We will illustrate, by means of the more familiar example of form factors, that associating a generic $F(r)$ with the internal structure of the particle leads to a self-consistent picture and does not create any visible problems even in the case of such a light particle as the pion, for which the relativistic interlace of c.m. and internal variables is maximal. To be specific, we will consider form factors calculated in the framework of ChPT, mainly for two reasons. First, we want to discuss the generalized polarizabilities $F(q^2)$ predicted by the *same* theory in order to check that our consideration of polarizabilities at scales $r \sim 1/m_\pi$ is not in conflict with other observables. Second, at present ChPT is the best tool for describing hadron structure at large scales and it is the only theory which agrees with the recent MAMI data on generalized polarizabilities of the proton [11].

We would like to mention the following technical aspect concerning the Fourier transformation of a q distribution obtained within ChPT. Such distributions are only reliably known for small momenta, $q = \mathcal{O}(m_\pi)$. Moreover, a straightforward integration over q does not exist, because the integrand typically diverges for large values of q. We therefore enforce convergence by introducing a cutoff Λ. Clearly, such a cutoff disturbs the corresponding r distributions at distances $r \lesssim 1/\Lambda$ which are beyond the scope of ChPT. On the other hand, one might expect that the results are cutoff independent when $r \gg 1/\Lambda$. In practice, we make use of a Gaussian cutoff and, for any q-dependent form factor or polarizability $F(q^2)$, we calculate $F(r)$ as

$$F(r) = \lim_{\Lambda\to\infty} 4\pi \int_0^\infty F(-Q^2)\,\frac{\sin(Qr)}{Qr}\,\exp\Big(-\frac{Q^2}{\Lambda^2}\Big)\,\frac{Q^2\,dQ}{(2\pi)^3}. \tag{5.2}$$

Depending on how small r is, we have to choose Λ large enough in order to approach the limit of $\Lambda = \infty$. In particular, for all generalized polarizabilities considered below we have found

the regularized Fourier integral of Eq. (5.2) to be independent of Λ for $\Lambda \geq 30$ GeV within an accuracy better than 2% even at distances as short as $r = 0.1$ fm. Stated differently, the cutoff $\Lambda = 30$ GeV is sufficient to resolve spatial distributions of polarizations at scales $r \sim 0.1$ fm. In the case of electromagnetic form factors, having steeper spatial distributions (see below), the 30 GeV cutoff is sufficient for a good resolution up to distances of $r \sim 0.2$ fm.

There is yet another way of calculating the Fourier integral of Eq. (5.2) based on a contour deformation in the complex Q plane. This method is applicable when the analytical continuation of $F(q^2)$ to time-like momenta is known as in the case of the ChPT predictions. Since all singularities of $F(t)$ are located at real positive t, it is possible to write a dispersion relation for F,[12]

$$F(q^2) = \frac{1}{\pi} \int_{t_{\min}}^{\infty} \operatorname{Im} F(t) \, \frac{dt}{t - q^2 - i0^+} \tag{5.3}$$

which allows one to recast the Fourier integral for $F(r)$ at $r > 0$ as a superposition of Yukawa functions:

$$F(r) = \frac{1}{4\pi^2 r} \int_{t_{\min}}^{\infty} e^{-r\sqrt{t}} \operatorname{Im} F(t) \, dt. \tag{5.4}$$

We made use of both methods and arrived at identical results for $F(r)$.

It is worthwhile recalling that polynomial pieces in $F(q^2)$ generate $\delta(r)$ terms or derivatives thereof in the Fourier transform $F(r)$. Such terms typically originate from higher-order counter terms in the Lagrangian which are multiplied by a priori unknown low-energy constants. However, in the Fourier transform, they do not contribute to $F(r)$ at finite $r \neq 0$. In other words, the Fourier integral acts as a filter which only transmits genuine effects of pion (or kaon) loops through their contributions to a nonpolynomial part of $F(q^2)$ and to $\operatorname{Im} F(q^2)$, respectively.

B. Form factors

As a first illustration, we briefly discuss the scalar and vector form factors of the pion as obtained in two-flavor ChPT in the limit of isospin symmetry. These form factors parametrize matrix elements of the scalar density $S(x) \equiv \hat{m}[\bar{u}(x)u(x) + \bar{d}(x)d(x)]$ with $\hat{m} = m_u = m_d$ and the isovector electromagnetic current $j_\mu^V(x) \equiv \frac{1}{2}\bar{q}(x)\tau_3\gamma_\mu q(x)$, respectively:

$$\begin{aligned} \langle \pi(p')|S(0)|\pi(p)\rangle &= F_S(q^2), \\ \langle \pi^+(p')|j_\mu^V(0)|\pi^+(p)\rangle &= (p+p')_\mu F_V(q^2), \qquad q = p' - p. \end{aligned} \tag{5.5}$$

---

[12] Additional subtractions may be required resulting in additional polynomial contributions.

Recently, the one-loop calculations of $F_S(q^2)$ and $F_V(q^2)$ by Gasser and Leutwyler [41] have been extended to the two-loop level $\mathcal{O}(p^6)$ [42,43].

In order to simplify the discussion, we perform two subtractions in the form factors $F(t)$ and plot subtracted (and normalized) functions,

$$\bar{F}(t) = \frac{1}{F(0)}\Big[F(t) - tF'(0) - \frac{1}{2}t^2 F''(0)\Big]. \tag{5.6}$$

By that means we avoid polynomial contributions of $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$, which depend on low-energy constants, and emphasize the pieces originating from pion loops.[13] As stated before, such subtractions are not visible through the Fourier filter at $r > 0$ and are thus irrelevant for the determination of $F(r)$.

At $\mathcal{O}(p^4)$, the subtracted scalar form factor of the pion reads

$$\bar{F}_S(q^2) = 1 - \frac{m_\pi^2}{16\pi^2 F_\pi^2}\left[\frac{2x-1}{2}J^{(0)}(x) + \frac{19x^2 - 10x}{120}\right], \tag{5.7}$$

where $x = q^2/m_\pi^2$, and the function $J^{(0)}(x)$ is defined as[14]

$$J^{(0)}(x) = \int_0^1 \ln[1 + x(y^2 - y) - i0^+]\, dy \;=\; -2 - \sigma \ln\Big(\frac{\sigma - 1}{\sigma + 1}\Big), \quad \sigma \equiv \sqrt{1 - \frac{4}{x}}, \quad x < 0. \tag{5.8}$$

As numerical values, we use $F_\pi = 92.4$ MeV [47] and the charged pion mass $m_\pi = 139.6$ MeV. The polynomial in Eq. (5.7) results in vanishing first and second derivatives of $\bar{F}_S$ at $t = 0$.

At one-loop order, the subtracted vector form factor is of a similar form:

$$\bar{F}_V(q^2) = 1 - \frac{m_\pi^2}{16\pi^2 F_\pi^2}\left(\frac{x-4}{6}J^{(0)}(x) + \frac{3x^2 - 20x}{180}\right). \tag{5.9}$$

At the two-loop level, the scalar and vector form factors are given by more lengthy expressions which can be found in Refs. [42,43]. To be specific, we made use of Eqs. (3.6)–(3.8) and (3.16)–(3.18) of Ref. [43], using the parameters (low-energy constants) $\bar{l}_1 = -1.7$, $\bar{l}_2 = 6.1$, $\bar{l}_3 = 2.9$, $\bar{l}_4 = 4.472$, $\bar{l}_6 = 16.0$ (set I [43]).

[13] Alternatively, we could keep the polynomial contribution of the pion loops. However, in that case the result would still depend on the renormalization condition chosen.

[14] The results for $0 \leq x < 4$ and $4 < x$ are obtained by analytical continuation.

For comparison, we discuss as another example the isovector electric form factor of the nucleon, $G_E^V(q^2)$, to leading nontrivial order [$\mathcal{O}(p^3)$] within HBChPT [44,45]. With the above two subtractions one obtains[15]

$$\bar{G}_E^V(q^2) = 1 - \frac{m_\pi^2}{16\pi^2 F_\pi^2}\left(\frac{x-4}{6}J^{(0)}(x) + \frac{3x^2-20x}{180}\right) - \frac{m_\pi^2 g_A^2}{16\pi^2 F_\pi^2}\left(\frac{5x-8}{6}J^{(0)}(x) + \frac{21x^2-40x}{180}\right), \tag{5.10}$$

where $g_A = 1.267$ is the axial-vector coupling constant.

The scalar form factor of the nucleon and the corresponding spatial distribution were recently discussed by Robilotta [46] in the context of the two-pion-exchange contribution to the NN potential (see Fig. 8 of that reference).

In Fig. 1 we show the three form factors $\bar{F}(q^2)$ of Eqs. (5.7), (5.9), and (5.10) and their corresponding Fourier transforms $F(r)$ which, in the case of $F = F_V$ and $F = G_E^V$, are interpreted as the electric charge density.[16] For the sake of comparison we also present the results of a pole approximation to those form factors,

$$F(q^2) = \frac{m^2}{m^2 - q^2}, \qquad 4\pi r^2 F(r) = m^2\, r\, e^{-mr}, \tag{5.11}$$

with the mass scale $m = 770$ MeV arbitrarily chosen to be the ρ-meson mass.

As a first observation, we note that all spatial distributions considered are positive at the one-loop order. Such a positive sign confirms a (maybe too) naive understanding of the peripheral structure of the target as being created by a cloud of virtual pions which create, in the case of a π^+ or proton target, a positive electric charge density of finite size. Also the scalar density seen by an external scalar field carries a positive sign. The QCD coupling of an external scalar field $\tilde{s}(x)$ to the scalar density $S(x)$,

$$\mathcal{L}_{\rm ext} = -\hat{m}[\bar{u}(x)u(x) + \bar{d}(x)d(x)]\tilde{s}(x), \tag{5.12}$$

is described through the Lagrangian $\frac{1}{4}F^2\mathrm{Tr}\,(\chi U^\dagger + U\chi^\dagger)$ at lowest order in ChPT, where, in the present case, $\chi = m_\pi^2 \tilde{s}(x)$. Inserting $U = (\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi})/F$, the lowest-order interaction reads

$$\mathcal{L}_{\rm ext}^{\rm eff} = -\frac{1}{2}m_\pi^2 \boldsymbol{\pi}^2(x)\tilde{s}(x) \equiv -S^{\rm eff}(x)\tilde{s}(x). \tag{5.13}$$

---

[15] In this case one subtraction would be sufficient to remove low-energy constants. Note that the functions $J(q^2)$ and $\zeta(q^2)$ used in Ref. [44] are related with the function $J^{(0)}(x)$, Eq. (5.8), by $16\pi^2\zeta(q^2) = -J^{(0)}(x)$ and $(96\pi^2/m_\pi^2)J(q^2) = -2x/3 + (x-4)J^{(0)}(x)$.

[16] Analytical representations of $F(r)$ in ChPT at one-loop order are given in Appendix A.

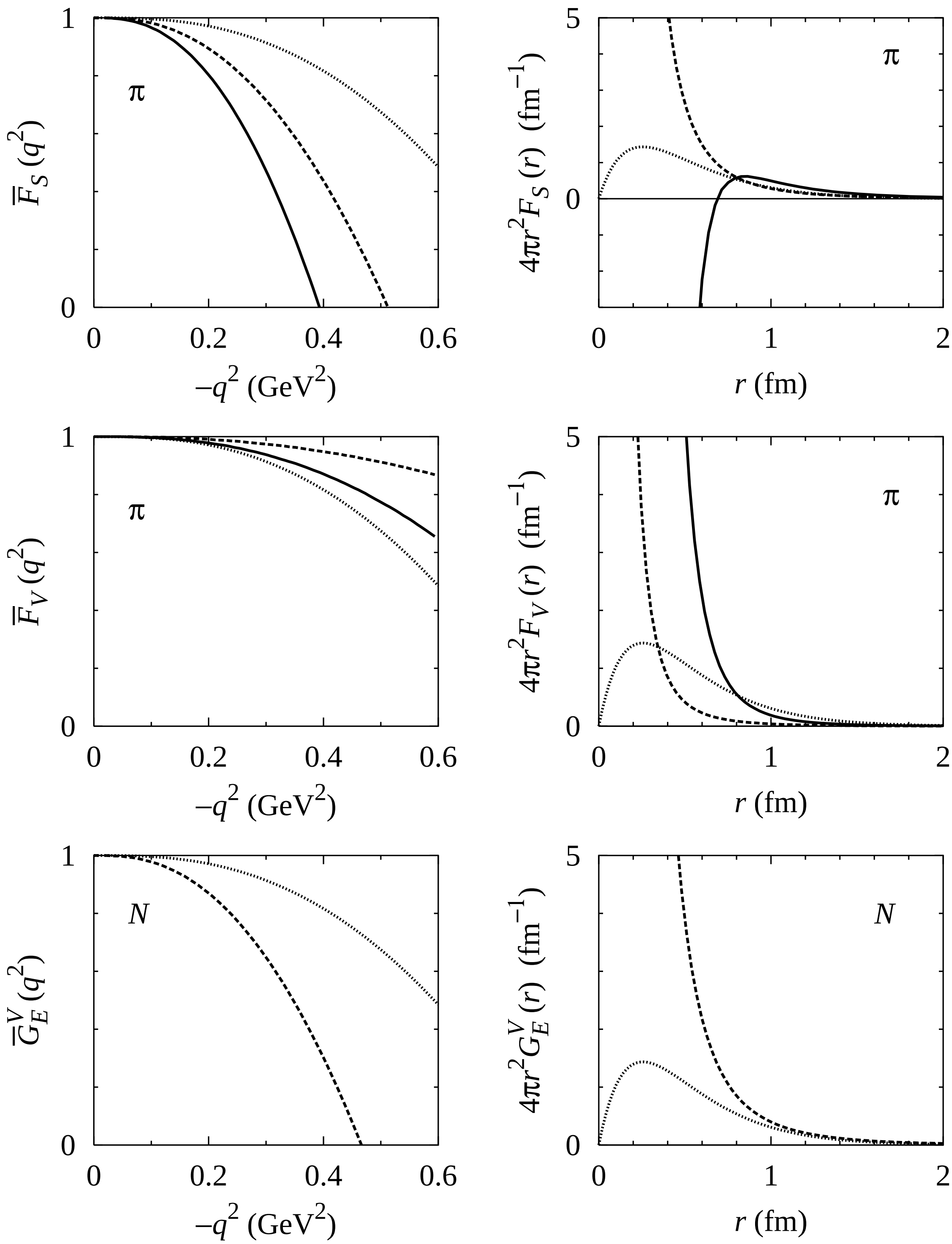

FIG. 1. Left panels: the (twice-subtracted) scalar and vector form factors of the pion in ChPT at order $\mathcal{O}(p^6)$ [43] and the isovector charge form factor of the nucleon in HBChPT at order $\mathcal{O}(p^3)$ [44,45]. Right panels: corresponding spatial distributions obtained as Fourier transforms. Dashed lines: one-loop prediction. Solid lines: two-loop prediction (for pions only). Dotted lines: prediction of the pole dominance model due to a scalar or vector meson of mass 770 MeV.

Thus, $S^{\rm eff}(x)$ has manifestly positive matrix elements which, when probed through $\tilde{s}$ as part of the pion cloud, lead to a positive density.

When two-loop corrections are taken into account, the spatial distributions at small distances, $r \lesssim 1$ fm, change drastically. In particular, the charge density $F_V(r)$ of the pion which at the one-loop level was concentrated at $r \lesssim 0.3$ fm, now extends up to $r \sim 0.6$ fm. An even more dramatic effect is observed for the scalar density $F_S(r)$, which due to the two-loop contribution changes sign at $r \lesssim 0.7$ fm. However, one should keep in mind that ChPT is a low-energy effective field theory and it is likely that higher-order corrections will change the spatial distributions at short distances substantially. An indication for such a scenario is given by the dotted lines in Fig. 1 which refer to short-range mechanisms such as vector mesons. Of course, such mechanisms are only included in ChPT by means of low-energy couplings in the effective Lagrangian.

Finally, let us emphasize again that the above examples do not pretend to *prove* that spatial distributions $F(r)$ have a strict operational sense. However, they certainly suggest a simple intuitive picture of the hadron periphery which is the natural domain of ChPT.

C. Polarizabilities

We now extend the discussion to spatial distributions associated with the generalized polarizabilities as introduced in Sec. IV. The predictions for the generalized dipole polarizabilities $\alpha_L(q^2)$ and $\beta(q^2)$ of the nucleon in the framework of SU(2)$_f$ HBChPT at $\mathcal{O}(p^3)$ read [21,48]

$$
\begin{aligned}
\alpha_L(q^2) &= E_\pi^2 \int_0^1 [8s^2 + s m_\pi^2(16y^2 - 16y + 9) - 3m_\pi^4(2y-1)^2]\, \frac{dy}{32 s^{5/2}} \\
&= \frac{E_\pi^2}{8m_\pi a(a+1)}\Big[(a+1)(2a-1)\frac{\arctan\sqrt{a}}{\sqrt{a}} + 2a + 1\Big], \\
\beta(q^2) &= E_\pi^2 \int_0^1 (2y-1)^2 (s+m_\pi^2)(4s - 3m_\pi^2)\, \frac{dy}{16 s^{5/2}} \\
&= \frac{E_\pi^2}{16 m_\pi a(a+1)}\Big[(a+1)(2a+1)\frac{\arctan\sqrt{a}}{\sqrt{a}} - 2a - 1\Big],
\end{aligned}
\tag{5.14}
$$

where $s = m_\pi^2 + q^2(y^2 - y)$, $a = -q^2/(4m_\pi^2)$, and $E_\pi = e g_A \sqrt{2}/(8\pi F_\pi)$ is the Kroll-Ruderman amplitude of $\pi^\pm$ photoproduction in the chiral limit.[17]

---

[17] The integral representations of Eq. (5.14) are, of course, less convenient than the equivalent elementary formulas. They are given here only as a historical reference, in the form in which they were first calculated and reported [48]. The same results were found independently by the authors of Refs. [19–21]. Recently, results have also been given in the framework of the Small Scale Expansion including the Δ-isobar as a dynamic degree of freedom [21].

The transverse electric polarizability of the nucleon remains yet to be determined.

There are no published calculations of the nucleon's generalized polarizabilities $\alpha_L(q^2)$ and $\beta(q^2)$ within $\mathrm{SU}(3)_f$ HBChPT, except for the case $q^2 = 0$ considered in Refs. [49,50]. However, it is straightforward to extend the results of Eq. (5.14) to the $\mathrm{SU}(3)_f$ case in the limit of equal N, Λ, and Σ masses. In this limit, the *square* baryon-octet mass differences is considered as small in comparison with the square kaon mass, m_K^2, which empirically is a good approximation. Then, the structure of Feynman diagrams contributing to Compton scattering in the $\mathrm{SU}(3)_f$ and $\mathrm{SU}(2)_f$ cases, respectively, is very similar, the only difference resulting (a) from different meson-baryon couplings for charged mesons (or, stated differently, from different Kroll-Ruderman amplitudes) and (b) from different masses of the Goldstone bosons entering the appropriate loop diagrams. The generalized polarizabilities of the nucleon in $\mathrm{SU}(3)_f$ HBChPT receive, in addition to the results of Eqs. (5.14), contributions due to kaon loops which are given by the same expressions as in Eqs. (5.14) after the replacements $m_\pi \to m_K$ and $E_\pi \to E_K$, where

$$
\begin{aligned}
E_\pi^2 &\equiv E^2_{\mathrm{KR}}(\gamma N \to \pi^\pm N) = \Big(\frac{e\sqrt{2}}{8\pi F_\pi}\Big)^2 I_\pi\,, \\
E_K^2 &\equiv E^2_{\mathrm{KR}}(\gamma N \to K^+\Lambda) + E^2_{\mathrm{KR}}(\gamma N \to K^+\Sigma) = \Big(\frac{e\sqrt{2}}{8\pi F_K}\Big)^2 I_K
\end{aligned}
\tag{5.15}
$$

and

$$
I_\pi = g_A^2 = (D+F)^2, \qquad I_K = \begin{cases} \frac{2}{3}D^2 + 2F^2, & \text{proton} \\ (D-F)^2, & \text{neutron} \end{cases}
\tag{5.16}
$$

(cf. [49,50]). In an $\mathcal{O}(p^3)$ calculation, the difference between the pion and kaon decay constants is of higher order. Empirically, $F_K \simeq 1.22 F_\pi$ [47], but in our numerical analysis we make use of a universal value $F_K = F_\pi$ with $F_\pi = 92.4$ MeV. Furthermore, we insert the $\mathrm{SU}(3)_f$ ratio $F/D = 2/3$.

By applying the same procedure to the other octet baryons, one obtains a generic polarizability $F(q^2)$ as

$$
F(q^2) = \Big[\text{Eq. (5.14)}\Big] \times \frac{I_\pi}{g_A^2} \; + \; \Big[\text{Eq. (5.14)}\Big]_{m_\pi \to m_K} \times \frac{I_K}{g_A^2}.
\tag{5.17}
$$

The $\mathrm{SU}(3)_f$ coefficients I_π and I_K are identical with the ones given in Eq. (4) of Ref. [49] for the case $q^2 = 0$. For convenience, we collect these coefficients in Table I (using $F/D = 2/3$). Note, however, that the pion-loop contribution for Λ and $\Sigma^\pm$ is perhaps not given very accurately by this procedure, since the baryon mass difference in the transitions $\pi^\pm\Lambda \leftrightarrow \Sigma^\pm$ is not fully negligible in comparison with the pion mass m_π.

At $\mathcal{O}(p^4)$, $\mathrm{SU}(2)_f$ and $\mathrm{SU}(3)_f$ ChPT predictions for all three dipole polarizabilities $\alpha_L(q^2)$, $\alpha_T(q^2)$, and $\beta(q^2)$ have been reported for pions and kaons [27,28]. At that order, the three generalized dipole polarizabilities are degenerate, i.e.

TABLE I. Flavor coefficients I_π and I_K determining pion and kaon loop contributions to the generalized polarizabilities of octet baryons in $\mathrm{SU}(3)_f$ HBChPT at order $\mathcal{O}(p^3)$.

B	I_π	I_π/g_A^2	I_K	I_K/g_A^2
p	$(D+F)^2$	1.00	$\frac{2}{3}D^2+2F^2$	0.56
n	$(D+F)^2$	1.00	$(D-F)^2$	0.04
Λ	$\frac{4}{3}D^2$	0.48	$\frac{1}{3}D^2+3F^2$	0.60
Σ^+	$\frac{2}{3}D^2+2F^2$	0.56	$(D+F)^2$	1.00
Σ^0	$4F^2$	0.64	D^2+F^2	0.52
Σ^-	$\frac{2}{3}D^2+2F^2$	0.56	$(D-F)^2$	0.04
Ξ^0	$(D-F)^2$	0.04	$(D+F)^2$	1.00
Ξ^-	$(D-F)^2$	0.04	$\frac{2}{3}D^2+2F^2$	0.56

$$\alpha_L(q^2)=\alpha_T(q^2)=-\beta(q^2) \quad \text{and} \quad \alpha_L(r)=\alpha_T(r)=-\beta(r), \tag{5.18}$$

and hence we only need to discuss, say, the generalized magnetic polarizability which can be expressed as [28]

$$-\beta(q^2)=\frac{e^2}{4\pi M}\,\frac{1}{(4\pi F_\pi)^2}\Big[A+(B_\pi+C_\pi x_\pi)J^{(0)'}(x_\pi)+C_K x_K J^{(0)'}(x_K)\Big]. \tag{5.19}$$

Here, M is the mass of the particle in question,

$$x_\pi=\frac{q^2}{m_\pi^2}\,, \quad x_K=\frac{q^2}{m_K^2}\,, \tag{5.20}$$

the function $J^{(0)'}(x)$ is given by

$$J^{(0)'}(x)=\frac{J^{(0)}(x)}{dx}=\frac{2J^{(0)}(x)+x}{x(x-4)}, \tag{5.21}$$

and A, B_π, C_π, and C_K are constants given below. The terms in Eq. (5.19) depending on x_π and x_K represent contributions of pion and kaon loops, respectively. The q^2-independent term proportional to A originates from a contribution at short distances represented by low-energy constants in the effective chiral Lagrangian [51]:

$$\begin{aligned} &A(\pi^\pm)=A(K^\pm)=64\pi^2(L_9^r+L_{10}^r)=\frac{2F_A}{F_V}=0.90\pm 0.12,\\ &A(\pi^0)=A(K^0)=A(\bar{K}^0)=0. \end{aligned} \tag{5.22}$$

The numerical value of $A(\pi^\pm)$ is fixed [52] by the experimentally known axial (F_A) and theoretically known vector (F_V) form factors of the radiative pion decay $\pi^+\to e^+\nu_e\gamma$ using $F_A/F_V=0.448\pm 0.062$ [47]. The relation $A(\pi)=A(K)$ is valid in the $\mathrm{SU}(3)_f$-symmetry

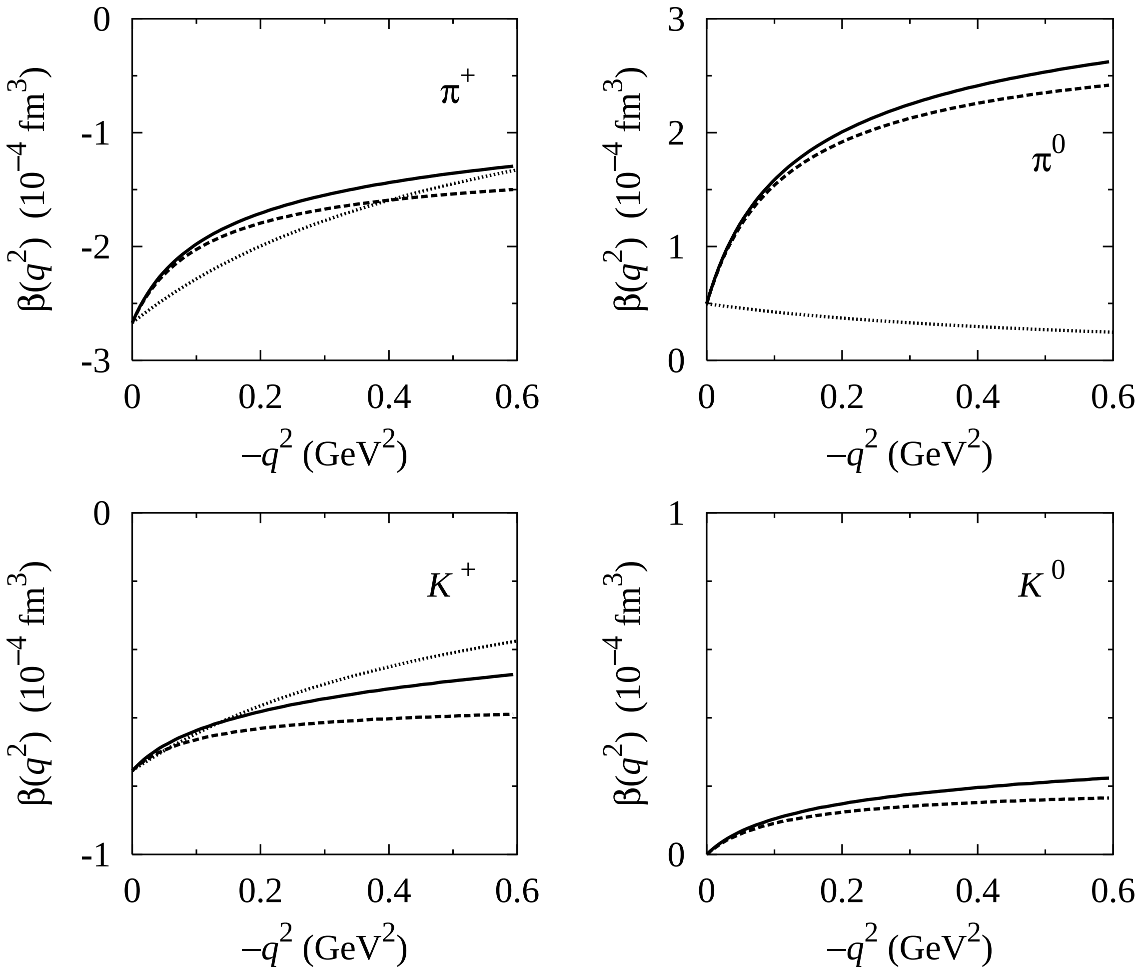

FIG. 2. Generalized magnetic polarizability $\beta(q^2)$ of pions and kaons at $\mathcal{O}(p^4)$ [27,28]. Dashed lines: contribution of pion loops. Solid lines: contribution of pion and kaon loops. Dotted lines: VMD predictions normalized to $\bar{\beta}$ as given by $\mathrm{SU}(3)_f$ ChPT.

limit. The other constants entering Eq. (5.19) are flavor-dependent coefficients which determine contributions of pion and kaon loops, respectively:

$$C_\pi = \begin{cases} -1 & \text{for } \pi^0, \\ -1/2 & \text{for } \pi^\pm, \\ -1/4 & \text{for kaons,} \end{cases} \qquad B_\pi = \begin{cases} 1 & \text{for } \pi^0, \\ 0 & \text{otherwise,} \end{cases} \qquad C_K = \begin{cases} -1/2 & \text{for } K^\pm, \\ -1/4 & \text{otherwise.} \end{cases} \tag{5.23}$$

The generalized polarizability of the Goldstone bosons are shown together with the results for the proton, the Σ^-, and the Ξ^- in Figs. 2 and 3.

The spatial distributions calculated as the Fourier transforms of the generalized polarizabilities are shown in Figs. 4 and 5. The corresponding analytical results obtained through Eq. (5.4) are given in Appendix A. For large r, the pion and kaon loop contributions to a generic polarizability $F(r)$ follow an exponential behavior $e^{-2m_\pi r}$ and $e^{-2m_K r}$, respectively, as determined by the nearest singularities of $F(q^2)$ for time-like momenta, $q^2 = 4m_\pi^2$ and

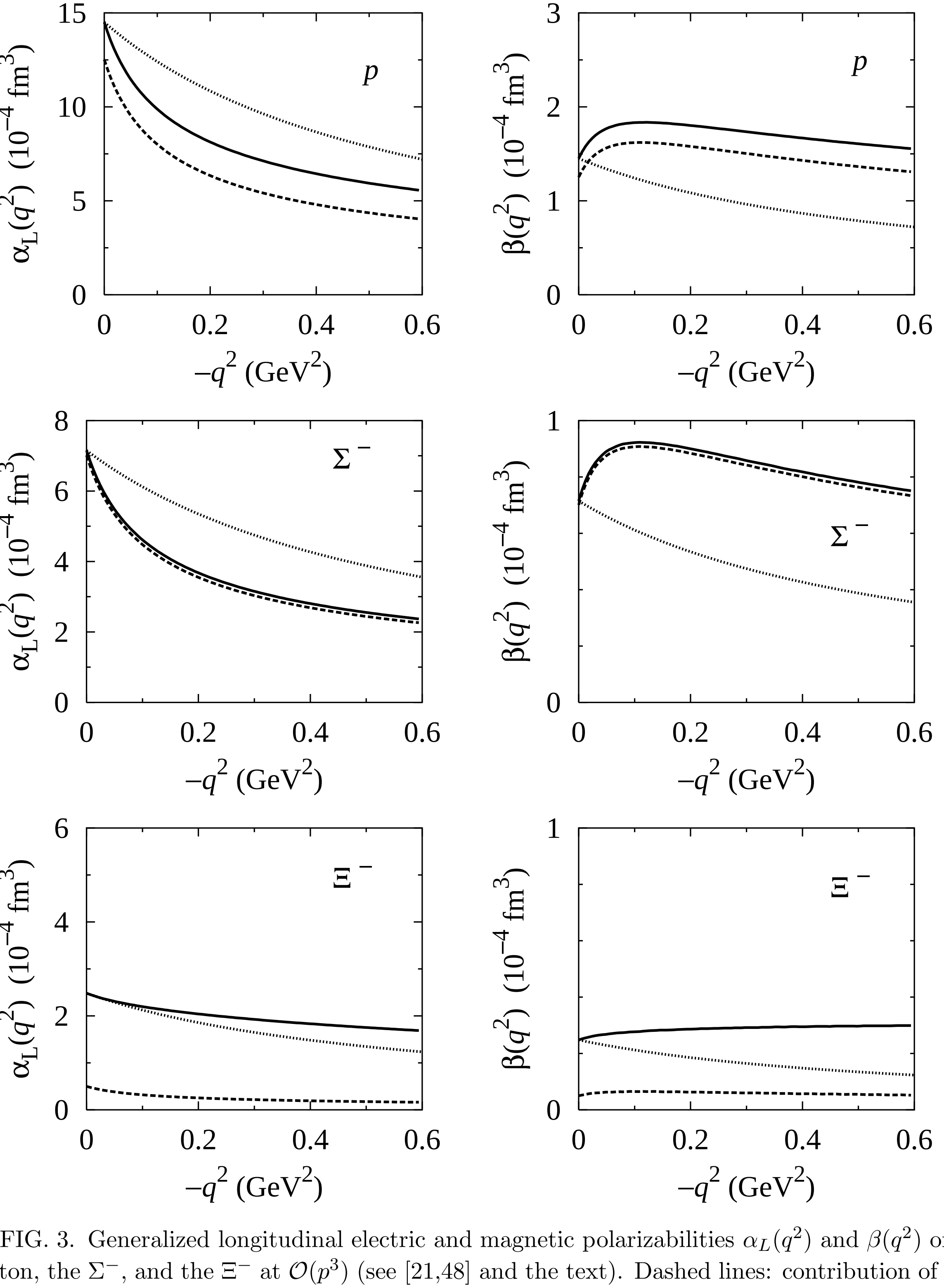

FIG. 3. Generalized longitudinal electric and magnetic polarizabilities $\alpha_L(q^2)$ and $\beta(q^2)$ of the proton, the Σ^-, and the Ξ^- at $\mathcal{O}(p^3)$ (see [21,48] and the text). Dashed lines: contribution of pion loops. Solid lines: contribution of pion and kaon loops. Dotted lines: VMD predictions normalized to $\bar{\alpha}$ and $\bar{\beta}$ as given by SU(3)$_f$ ChPT.

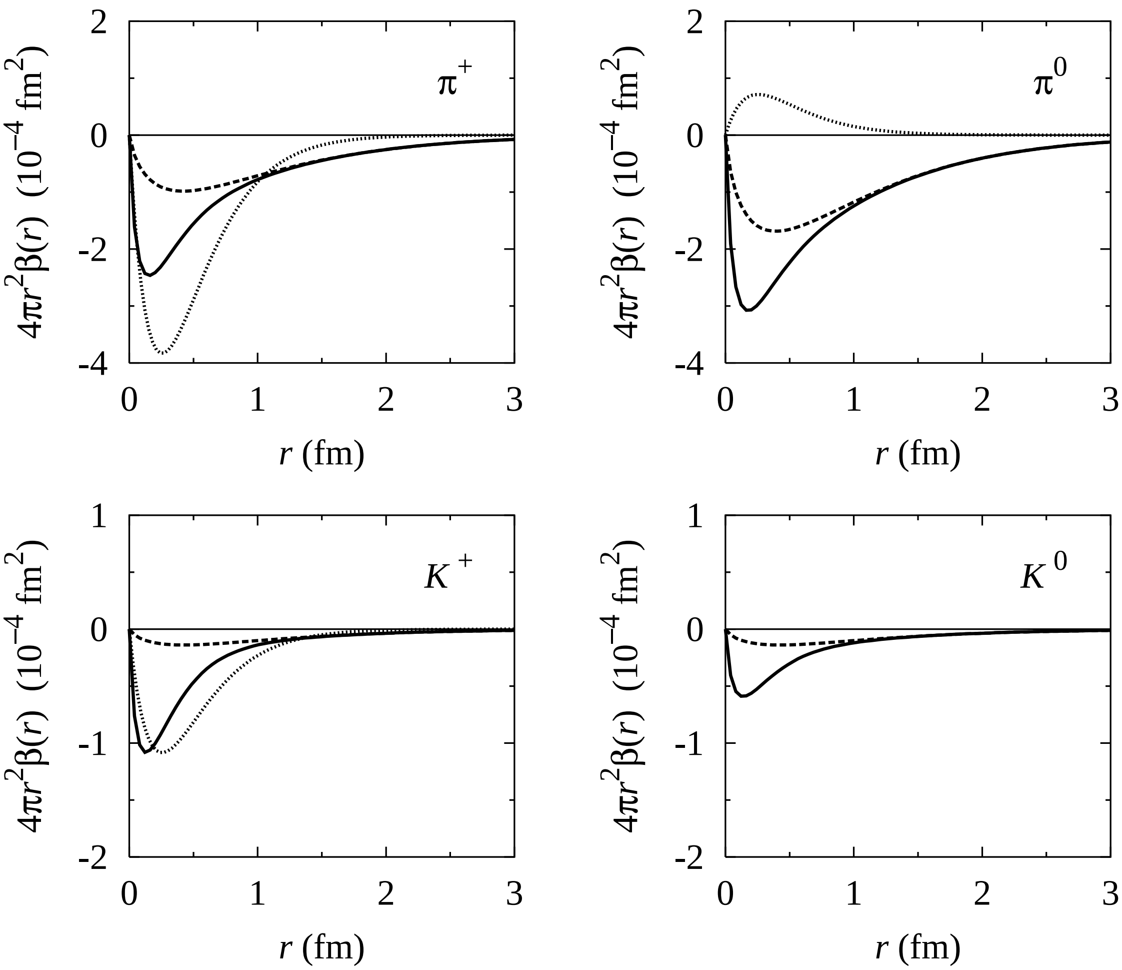

FIG. 4. Density of the magnetic polarizability $\beta(r)$ of pions and kaons at $\mathcal{O}(p^4)$. The δ-singularity at $r = 0$ is not shown (see the discussion of Eq. (5.24) in the text). Dashed lines: contribution of pion loops. Solid lines: contribution of pion and kaon loops. Dotted lines: VMD predictions normalized to $\bar{\beta}$ as given by $\mathrm{SU}(3)_f$ ChPT.

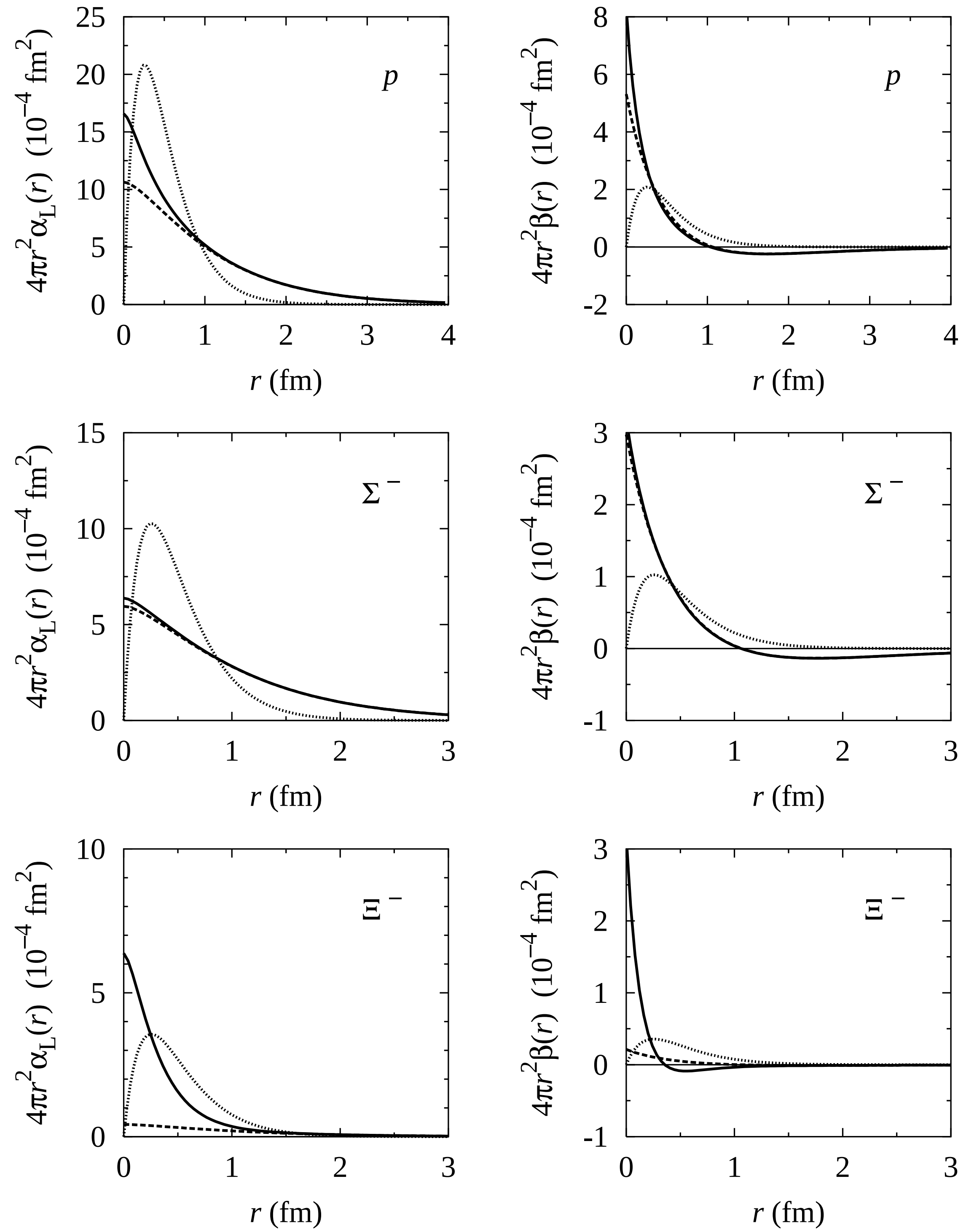

FIG. 5. Density of the longitudinal electric and magnetic polarizabilities $\alpha_L(r)$ and $\beta(r)$ of the proton, the Σ^-, and the Ξ^- at $\mathcal{O}(p^3)$. Dashed lines: contribution of pion loops. Solid lines: contribution of pion and kaon loops. Dotted lines: VMD predictions normalized to $\bar{\alpha}$ and $\bar{\beta}$ as given by $\mathrm{SU}(3)_f$ ChPT.

$q^2 = 4m_K^2$. The δ-singularity at $r = 0$ cannot be seen in these plots. However, at least for mesons, such a singularity exists for sure within ChPT, and it is determined by the asymptotic value of the polarizability for $q^2 \to -\infty$. Thus, the integrals of the spatial distributions over $r > 0$ are

$$\lim_{\epsilon \to 0+} \int_{\epsilon}^{\infty} 4\pi r^2 \beta(r)\, dr \;=\; \beta(q^2 = 0) - \beta(q^2 = -\infty). \tag{5.24}$$

The generalized polarizabilities of the octet baryons, given by Eqs. (5.14) and (5.19), vanish at infinity, so that the integral (5.24) gives just $\bar{\beta}$ in this case. This is not the case for mesons, since $x{J^{(0)}}'(x) \to 1$ for $x \to -\infty$. The corresponding δ-singularity in $4\pi r^2 \beta(r)$ is driven by the limits [53]

$$\begin{aligned}
\lim_{q^2 \to -\infty} \beta_{\pi^\pm}(q^2) &= \bar{\beta}_{\pi^\pm} + \frac{9}{2}\bar{\beta}_{\pi^0}, \\
\lim_{q^2 \to -\infty} \beta_{\pi^0}(q^2) &= \frac{15}{2}\bar{\beta}_{\pi^0}, \\
\lim_{q^2 \to -\infty} \beta_{K^\pm}(q^2) &= \bar{\beta}_{K^\pm} + \frac{9}{2}\frac{m_\pi}{m_K}\bar{\beta}_{\pi^0}, \\
\lim_{q^2 \to -\infty} \beta_{K^0}(q^2) &= 3\frac{m_\pi}{m_K}\bar{\beta}_{\pi^0}.
\end{aligned} \tag{5.25}$$

The SU$(2)_f$ results for pions are found in Eqs. (39) and (40) of Ref. [27].

It is very interesting that the loop contributions behave exactly as one would expect from a classical interpretation of the Langevin diamagnetism. In such a picture, a change in the external magnetic field would produce circulating currents induced in the charge density of the meson cloud which on their part give rise to an induced magnetization. Both pion and kaon clouds are seen to be diamagnetic, at least at large distances $r \gtrsim 1/m_\pi$. Simultaneously, these clouds generate a positive sign for the electric polarizability. However, in the case of the nucleon the diamagnetic character of the pion cloud disappears at distances $r \lesssim 1/m_\pi$, where paramagnetism prevails and $\beta(r)$ becomes positive.

When a hadron is probed by photons of very small (space-like) momenta $|\boldsymbol{q}| \ll m_\pi$, the magnetic response is essentially only sensitive to $\beta(q^2 = 0) = \bar{\beta}$. When the momentum $|\boldsymbol{q}|$ increases, the very peripheral, negative part of the magnetic polarizability of the pion cloud no longer contributes to

$$\beta(q^2) = \int \beta(r) \exp(i\boldsymbol{q} \cdot \boldsymbol{r})\, d\boldsymbol{r}, \tag{5.26}$$

due to the oscillatory behavior of the integrand for large distances. This explains why the magnetic polarizability $\beta(q^2)$ for *all* the particles (pions, kaons, nucleons, etc.) universally gets a positive increase (see Figs. 2 and 3). The slope of $\beta(q^2)$ at the origin is proportional to the mean square radius of the spatial distribution,

$$\frac{d\beta(0)}{dq^2} = \frac{2\pi}{3} \int_0^{\infty} r^4 \beta(r)\, dr. \tag{5.27}$$

Obviously, for all mesons, the cloud distribution $4\pi r^2\beta(r) \leq 0$ for all $r \geq 0$. Hence, the slope is negative as a function of q^2 (positive, when plotted against $-q^2$). Also, the curvature as a function of q^2 is concave. For the proton, the Σ^-, and the Ξ^- the asymptotic negative pion and kaon tails in the integral (5.27) dominate over the positive contribution coming from distances smaller than 1 fm, 1 fm, and 0.4 fm, respectively. This makes the slopes of the magnetic polarizability of all baryons considered positive as a function of $-q^2$ as well. On the other hand, such behavior is in some cases opposite to that expected from VMD. Suppose the q^2 dependence of $\beta(q^2)$ was determined by the ρ or ω mesons mediating electromagnetic interactions,

$$\Big[\beta(q^2)\Big]_{\rm VMD} = \bar\beta\,\frac{m_\rho^2}{m_\rho^2 - q^2}. \tag{5.28}$$

From this, the spatial distribution of $\beta(r)$

$$\Big[4\pi r^2\beta(r)\Big]_{\rm VMD} = m_\rho^2\, r\, e^{-m_\rho r}\,\bar\beta \tag{5.29}$$

would have the same the sign as $\bar\beta$ for all $r > 0$. For the π^0 and for the octet baryons, having positive $\bar\beta$, such a sign is in conflict with the ChPT behavior [see Figs. 3 and 5 in which the VMD distributions are also shown]. From a phenomenological point of view, the full magnetic polarizability $\bar\beta$ of the neutral pion should be approximately three times the $\mathcal{O}(p^4)$ prediction, mainly due to a paramagnetic contribution of the M1 transitions $\pi^0 \to \omega$ or ρ^0. In other words, a more realistic VMD curve should be three times higher than that shown in Fig. 4, so that the above conflict with ChPT would be even more severe.

In the case of the (longitudinal) electric polarizability of octet baryons the long-range, peripheral part is suppressed with increasing $-q^2$. This part is relatively large for all baryons except for the Ξ's, and thus $\alpha_L(q^2)$ shows a rapid decrease with increasing space-like q which is steeper than for VMD [see Fig. 3].

Another instructive feature of the plots shown in Figs. 4 and 5 is that the kaon-loop contribution is concentrated at about the same scale of r (or even at smaller scales) as the VMD contribution due to the large mass of kaon pairs, $2M_K \approx 1$ GeV. Kaon-loop contributions exactly fall into the short-range region where one finds other contributions of similar range treated via low-energy constants of the effective Lagrangian of ChPT. However, at $\mathcal{O}(p^3)$ no such counter terms contribute to the generalized polarizabilities. In other words, even though the relevant counter terms are formally of higher order in the power counting of ChPT, one may wonder whether their quantitative importance is underrated as compared to the kaon-loop contributions. Therefore, quantitative conclusions drawn from calculations keeping kaon loops and ignoring short-range contributions have to be treated with some care. For a similar conclusion, see Ref. [54].

The spatial distribution of $\beta(r)$ for the nucleon shown in Fig. 5 may also shed light on an old question which has remained open for more than 30 years: Why is the magnetic polarizability $\bar\beta$ of the proton so small ($\bar\beta_p \approx 2 \times 10^{-4}$ fm^3 [47]), despite of a very large paramagnetic contribution of the Δ resonance which, in various evaluations based on quark

models, dispersion theories, effective Lagrangians, etc., ranges between $+7$ to $+13\times10^{-4}$ fm^3 (see, for example, [55,56]). In this context, it is sometimes stated that the pion cloud produces a large diamagnetic (i.e. negative) contribution to $\bar{\beta}$ owing to the Langevin mechanism producing a negative magnetic susceptibility of bound charges. This point of view became especially popular after calculations of $\bar{\beta}_N$ in the Skyrme model (see, for example [57–59]), in which the pion field of the soliton produces indeed a very big diamagnetic susceptibility resulting from the pion-cloud tail of the soliton, ranging from -8 to -16×10^{-4} fm^3 [57–59]. Figure 5, however, shows that the pion-cloud periphery with $r \geq 1$ fm carries a very small diamagnetism of only -0.45×10^{-4} fm^3, where the last number is expected to be a reliable estimate because ChPT, even at leading nontrivial order, should be reasonable at such distances. We conclude that Skyrme models overestimate the pion-cloud contribution to diamagnetism and that a source for the missing diamagnetism is probably related with short-range mechanisms rather than with the pion cloud itself. In some dispersion theories of Compton scattering [60], an additional exchange with a hypothetical σ-meson is invoked in order to generate agreement with existing experimental data. This, of course, is just an oversimplified model for such a short-range contribution.

It would be very interesting and instructive to extend the presently available ChPT predictions for the generalized polarizabilities of the nucleon at least to order $\mathcal{O}(p^4)$. This would allow one to check whether a modification of the density $\beta(r)$, including relativistic (nucleon recoil) effects and other higher-order corrections, indeed provides sufficient diamagnetism [56].

VI. SUMMARY AND CONCLUSIONS

In the present paper we have developed a covariant formalism leading to a parametrization of the VCS tensor in terms of Lorentz invariants free from kinematical singularities and constraints. We motivated and performed a gauge-invariant division of the VCS tensor into contributions of generalized Born terms and a structure-dependent residual part. We then discussed the case of a real final photon and a space-like virtual initial photon which can be described in terms of three invariant amplitudes depending on three kinematical variables. In the limit $q' \to 0$, the three residual amplitudes reduce to functions of q^2 only which we identified as generalized dipole polarizabilities $\alpha_L(q^2)$, $\alpha_T(q^2)$, and $\beta(q^2)$. All of them can, in principle, be determined in virtual Compton scattering, although the transverse electric polarizability is not accessible in experiments sensitive to structure-dependent effects of $\mathcal{O}(q')$ only. We proposed a physical interpretation of these polarizabilities in terms of *spatial* distributions of an induced electric polarization and magnetization, respectively. In particular, we argued that a knowledge of all three polarizabilities is required for a full description of induced polarization phenomena. Following this line, we calculated spatial distributions for pions, kaons, and the baryon octet as Fourier integrals, using the predictions of ChPT. It was found that the distributions obtained confirmed expectations based on a picture of a hadron's periphery caused by a "classical" pion cloud. Of course, any practical *analysis* of experimental data on photon *scattering* will eventually deal with the original, precisely defined momentum-space form factors, polarizabilities, etc. Thus, one should handle the

found spatial distributions with care. On the other hand, the r-space interpretation of such q-dependent quantities clearly allows for a more intuitive visualization in analogy to the phenomenology and terminology of classical electrodynamics and nonrelativistic quantum mechanics.

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APPENDIX A: SPATIAL DISTRIBUTIONS TO ONE LOOP

In this appendix we collect the analytical results for the spatial distributions in ChPT at the one-loop level which are easily obtained through Eq. (5.4). In the formulas below we use the notation

$$t = q^2, \quad x_\pi = \frac{t}{m_\pi^2}, \quad x_K = \frac{t}{m_K^2}, \quad z_\pi = 2m_\pi r, \quad z_K = 2m_K r. \tag{A1}$$

The imaginary parts of the scalar and vector form factors of the pion, and of the isovector charge form factor of the nucleon are determined by the function $J^{(0)}(x)$ which has a branching point at $x = 4$:

$$\begin{aligned}
\operatorname{Im} F_S(t) &= F_S(0)\,\frac{2x_\pi - 1}{2} V(x_\pi), \\
\operatorname{Im} F_V(t) &= \frac{x_\pi - 4}{6} V(x_\pi), \\
\operatorname{Im} G_E^V(t) &= \Big(\frac{x_\pi - 4}{6} + g_A^2 \frac{5x_\pi - 8}{6}\Big) V(x_\pi),
\end{aligned} \tag{A2}$$

where

$$V(x) = \frac{\pi}{(4\pi F_\pi)^2} \sqrt{\frac{x-4}{x}}\, \theta(x-4). \tag{A3}$$

Evaluating the integral representation of Eq. (5.4) with the above imaginary parts results in

$$
\begin{aligned}
4\pi r^2 F_S(r) &= \frac{m_\pi^3 F_S(0)}{(4\pi F_\pi)^2}\Big[\frac{48}{z_\pi}\,K_0(z_\pi) + \frac{96+14z_\pi^2}{z_\pi^2}\,K_1(z_\pi)\Big],\\
4\pi r^2 F_V(r) &= \frac{m_\pi^3}{(4\pi F_\pi)^2}\Big[\frac{8}{z_\pi}\,K_0(z_\pi) + \frac{16}{z_\pi^2}\,K_1(z_\pi)\Big],\\
4\pi r^2 G_E^V(r) &= \frac{m_\pi^3}{(4\pi F_\pi)^2}\Big[\frac{8+40g_A^2}{z_\pi}\,K_0(z_\pi) + \frac{16+(80+8z_\pi^2)g_A^2}{z_\pi^2}\,K_1(z_\pi)\Big],
\end{aligned}
\tag{A4}
$$

where $K_\nu(z)$ is the modified Bessel function, $K_\nu(z) = \int_0^\infty e^{-z\cosh t}\,\cosh(\nu t)\,dt$.

The generalized polarizabilities of the nucleon, Eq. (5.14), at time-like momenta q have both a cut starting at $t = 4m_\pi^2$ and a pole at $t = 4m_\pi^2$:

$$
\begin{aligned}
\operatorname{Im}\alpha_L(t) &= \frac{\pi}{8}\,E_\pi^2\left[\Big(2+\frac{4m_\pi^2}{t}\Big)\frac{\theta(t-4m_\pi^2)}{\sqrt{t}} + 4m_\pi\,\delta(t-4m_\pi^2)\right],\\
\operatorname{Im}\beta(t) &= \frac{\pi}{16}\,E_\pi^2\left[\Big(2-\frac{4m_\pi^2}{t}\Big)\frac{\theta(t-4m_\pi^2)}{\sqrt{t}} - 4m_\pi\,\delta(t-4m_\pi^2)\right].
\end{aligned}
\tag{A5}
$$

Accordingly, Eq. (5.4) results in

$$
\begin{aligned}
4\pi r^2\alpha_L(r) &= \frac{E_\pi^2}{2}\Big[(1+z_\pi)e^{-z_\pi} + \frac{z_\pi^2}{2}\,\mathrm{Ei}(-z_\pi)\Big],\\
4\pi r^2\beta(r) &= \frac{E_\pi^2}{4}\Big[(1-z_\pi)e^{-z_\pi} - \frac{z_\pi^2}{2}\,\mathrm{Ei}(-z_\pi)\Big],
\end{aligned}
\tag{A6}
$$

where $\mathrm{Ei}(-z) = -\int_z^\infty (e^{-t}/t)\,dt$ is the exponential integral. The contribution of kaon loops is obtained by the substitutions $E_\pi \to E_K$ and $m_\pi \to m_K$ as explained in Section V.

For pions and kaons, the generalized polarizabilities of Eq. (5.19) have branching points at $t = 4m_\pi^2$ and $4m_K^2$ but no poles:

$$
\operatorname{Im}\beta(t) = \frac{e^2}{4\pi M}\,\frac{2\pi}{(4\pi F_\pi)^2}\,\left[\Big(\frac{B_\pi}{x_\pi} + C_\pi\Big)\frac{\theta(x_\pi-4)}{\sqrt{x_\pi^2-4x_\pi}} + C_K\frac{\theta(x_K-4)}{\sqrt{x_K^2-4x_K}}\right].
\tag{A7}
$$

Then Eq. (5.4) gives

$$
\begin{aligned}
4\pi r^2\beta(r) = \frac{e^2}{4\pi M}\,\frac{1}{(4\pi F_\pi)^2}\,\Big[&2m_\pi C_\pi z_\pi K_0(z_\pi) + 2m_K C_K z_K K_0(z_K)\\
&+ \frac{m_\pi}{2}\,B_\pi z_\pi^2\Big(K_1(z_\pi) - \int_{z_\pi}^\infty K_0(x)\,dx\Big)\Big].
\end{aligned}
\tag{A8}
$$

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